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Integral trees of diameter 4

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Abstract

An integral tree is a tree whose adjacency matrix has only integer eigenvalues. While most previous work by other authors has been focused either on the very restricted case of balanced trees or on finding trees with diameter as large as possible, we study integral trees of diameter 4. In particular, we characterize all diameter 4 integral trees of the form $T(m_1, t_1) \bullet T(m_2, t_2)$. In addition we give elegant parametric descriptions of infinite families of integral trees of the form $T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$ for any $n > 1$. We conjecture that we have found all such trees.

Keywords: Integral graphs, integral trees, balanced trees.

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1. Introduction

We will use G to denote a simple graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix of G will be denoted $A(G)$ and the characteristic polynomial of the adjacency matrix will be denoted $P(G)$. The graph G is said to be *integral* if all the roots to the characteristic equation are integers, that is all the eigenvalues of $A(G)$ are integers. In this paper we will focus only on trees.

A tree is called *balanced* if all the vertices at equal distance from the root have the same degree. It is standard to use $T(m, t)$ to denote the rooted diameter 4 tree where the root has m neighbours, and each of these vertices has t adjacent leaves. This tree is clearly balanced.

The first paper to consider integral trees appeared 30 years ago [16] in which Watanabe and Schwenk showed that $T(r, m)$, the balanced tree of diameter 4 is integral if and only if m and $r + m$ are squares. Almost all of the papers that have appeared since then have attempted to classify trees according to their diameter. Infinite families of integral trees of diameters 3, 4, 6 were found easily [1, 2, 7, 10, 11, 13, 14, 15, 17], and the goal was to find trees of diameter as large as possible. More recently several authors [12, 13, 15] have studied diameter 8 trees. Based on this work, in 2003 Hic and Pokorny [6] finally found integral trees of diameter 10, using extensive computer searches. Apart from the diameter 3 case, odd diameter trees seemed much harder to find. Some trees of diameter 5 were known early [9], but the first diameter 7 tree was not found until 2007 [4]. Most of the effort in these earlier papers dealt with trees that were balanced. This work was based on the foundation laid by Hic and Nedela in [5]. The Ph.D. thesis by Wang [18] is a good reference on this subject and contains much other interesting related material.

While diameter 3 trees have been completely characterized [1], diameter 4 trees already posed much more difficulty. Balanced trees were convenient as a stepping stone in the search for trees of larger and larger diameter. However, the focus on balanced trees meant that the vast majority of small diameter trees have so far been overlooked. The purpose of our work is to find the unbalanced trees of diameter 4.

An unbalanced tree of diameter 4 will be denoted $T(m; t_1, \ldots, t_m)$, indicating that the root has m neighbours and the i-th neighbour of the root has t_i leaves. The unbalanced tree $T(m; t_1, t_2)$ can be regarded as formed from the trees $T(m_1, t_1)$ and $T(m_2, t_2)$ where $m = m_1 + m_2$ by identifying their roots. This construction has been denoted by $T(m_1, t_1) \bullet T(m_2, t_2)$ and the reason for adopting this point of view is given by Theorem 2.3 below. Similarly the more general unbalanced trees of diameter 4 can equally be denoted $T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$. We show an example in Figure 1. We emphasize that all diameter 4 trees have this form, where at most one of the t_i may be 0. We note that some results on unbalanced diameter 4 trees have appeared in [19] and [8].

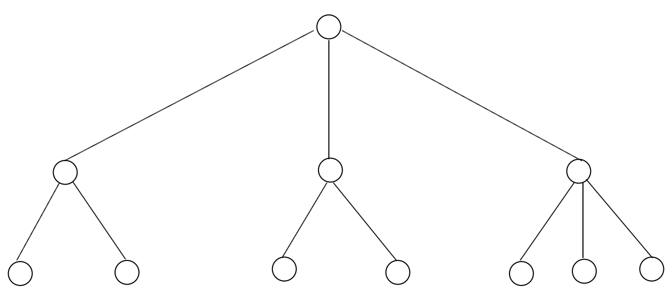

Figure 1: The tree $T(3; 2, 2, 3)$ or $T(2, 2) \bullet T(1, 3)$

Section 2 will describe the characteristic polynomials of the different trees used in this article. Section 3 will look at trees of the form $T(1, t_1) \bullet T(m, t_2)$ and we completely describe these trees. In section 4 the trees of type $T(m_1, t_1) \bullet T(m_2, t_2)$ where $m_i > 1$ will be examined. We will be able to give a complete characterization of these trees as well. The diameter 4 trees $T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$ will be studied in section 5 and we will be able to provide infinite families of such trees for every $n \geq 2$. We conjecture that we have found all solutions for the cases where all $m_i > 1$.

2. The characteristic polynomials of the trees

Since we will be needing the characteristic equations of the trees we study, we begin by summarizing the known results. Theorem 2.2 due to Watanabe and Schwenk [16] was the first substantial result in this area.

Lemma 2.1. [16] *The characteristic polynomial of $T(m,t)$ is*

$$P\left(T(m,t),x\right) \quad = \quad x^{m(t-1)+1}\left[x^2-t\right]^{m-1}\left[x^2-(t+m)\right]. \tag{1}$$

Theorem 2.2. [16] *$T(m,t)$ is integral if and only if t and $m+t$ are squares.*

The following theorem, due to Godsil and McKay in 1982, is the main tool that has enabled all the investigations into the spectra of trees of the type described in the Introduction. We rely on it heavily.

Theorem 2.3. [3] *Let G_1 and G_2 be graphs and let $u \in V(G_1)$ and $v \in V(G_2)$ then*

$$\begin{aligned} P(G_1 \bullet G_2, x) \quad = \quad & P(G_1,x)P(G_2-v,x)+P(G_1-u,x)P(G_2,x) \\ & -xP(G_1-u,x)P(G_2-v,x). \end{aligned} \tag{2}$$

Application of Theorem 2.3 yields the following result concerning the trees we will deal with. It can be viewed as a generalization of Lemma 2.1.

Corollary 2.4. *The characteristic polynomial of $T(m_1,t_1)\bullet\cdots\bullet T(m_n,t_n)$ is*

$$x^{1+\sum_{i=1}^{n} m_i(t_i-1)}\left[\prod_{i=1}^{n}\left(x^2-t_i\right)^{m_i}-\sum_{i=1}^{n} m_i\left(x^2-t_i\right)^{m_i-1}\prod_{j=1,j\neq i}^{n}\left(x^2-t_j\right)^{m_j}\right].$$

The following Corollary will be useful in what follows. Most of our characterizations will result in trees with rational values for the various parameters (m_i and t_i). This next result gives us the means to scale these rational values up to integer values, thus producing integral trees. The tree $T(m_1,t_1)\bullet\cdots\bullet T(m_n,t_n)$ is a *rational tree* if $m_i, t_i \in \mathbb{Q}$ and the eigenvalues are integers.

Corollary 2.5. *If $T(m_1,t_1)\bullet\cdots\bullet T(m_n,t_n)$ is a rational tree and t_i is a square for all i, then $T(d^2m_1,d^2t_1)\bullet\cdots\bullet T(d^2m_n,d^2t_n)$ is rational, where d is any positive integer.*

Proof. From Corollary 2.4 the characteristic polynomial of $T(m_1,t_1)\bullet\cdots\bullet T(m_n,t_n)$ is

$$x^{1+\sum_{i=1}^{n} m_i(t_i-1)} \prod_{i=1}^{n} \left(x^2 - t_i\right)^{m_i-1}$$
$$\left[\prod_{i=1}^{n} \left(x^2 - t_i\right) - \sum_{i=1}^{n} m_i \prod_{j=1, j\neq i}^{n} \left(x^2 - t_j\right)\right]$$
$$= x^{1+\sum_{i=1}^{n} m_i(t_i-1)} \prod_{i=1}^{n} \left(x^2 - t_i\right)^{m_i-1} \left[(x^2 - a_1^2)\dots(x^2 - a_n^2)\right].$$

Then the characteristic polynomial of $T(d^2m_1, d^2t_1) \bullet \cdots \bullet T(d^2m_n, d^2t_n)$ is

$$x^{1+\sum_{i=1}^{n} d^2m_i(d^2t_i-1)} \prod_{i=1}^{n} \left(x^2 - d^2t_i\right)^{d^2m_i-1}$$
$$\left[\prod_{i=1}^{n} \left(x^2 - d^2t_i\right) - \sum_{i=1}^{n} d^2m_i \prod_{j=1, j\neq i}^{n} \left(x^2 - d^2t_j\right)\right]$$
$$= x^{1+\sum_{i=1}^{n} d^2m_i(d^2t_i-1)} \prod_{i=1}^{n} \left(x^2 - d^2t_i\right)^{d^2m_i-1} \left[(x^2 - d^2a_1^2)\dots(x^2 - d^2a_n^2)\right]$$

and from Corollary 2.4, $T(d^2m_1, d^2t_1) \bullet \cdots \bullet T(d^2m_n, d^2t_n)$ is rational so long as the t_i are squares. □

3. Trees of type $T(1, t_1) \bullet T(m, t_2)$

In this section we will describe all integral trees of the form $T(1, t_1) \bullet T(m, t_2)$ where $m > 0$. We assume $t_1 \neq t_2$ so that we are not in the case covered by Theorem 2.2. These trees fall into two families, the first is a 2-parameter family and the second is a 3-parameter family.

We can easily show that $m > 1$. For if $m = 1$, calculating the characteristic equation by Corollary 2.4 tells us:

$$x^{m(t_2-1)+t_1} \left[(x^2 - t_1)(x^2 - t_2) - (x^2 - t_1) - (x^2 - t_2)\right] = 0.$$

We can rewrite this as

$$(x^2 - t_1 - 1)(x^2 - t_2 - 1) = 1$$

and since the factors on the left hand side are both equal to $+1$ or both equal to -1, we immediately see that $t_1 = t_2$. So this case is covered by Theorem 2.2, which then shows there are no such integral trees. Consequently, we can assume that $m > 1$.

Theorem 3.1. *Let $m > 1$ and $t_1 \neq t_2$. Then $T(1, t_1) \bullet T(m, t_2)$ is integral if and only if*

(1) For integer parameters a and b, then $m = -a(2b+1+a)$, $t_1 = b^2+2b+a$, $t_2 = (b+a)^2$ where $b > 0$ and $0 < -a < min\{(b+1)^2, 2b+1\}$ or $-b > 0$ and $0 < a < -2b-1$; or

(2) For integer parameters a_1, a_2, a_3 where $a_1 \neq 0$ and $a_3 \neq a_1a_2$, let $b = \frac{a_1a_2+a_3}{2} + \frac{a_1(a_1-1)(a_2+1)(a_2-1)}{2(a_3-a_1a_2)}$. If $b \in \mathbb{Z}\backslash\{0\}$ then $m = a_1(2ba_2 - a_2^2 - a_1)$, $t_1 = (b-a_2)^2 + a_1 - 1$, $t_2 = (b-a_3)^2$, so long as $m > 0$ and $t_1, t_2 \geq 0$.

Proof. From Corollary 2.4 the characteristic equation of $T(1, t_1) \bullet T(m, t_2)$ is

$$\begin{gathered} x^{-m+mt_2+t_1}(x^2-t_2)^{m-1}\times \\ \times\left[x^4-(t_1+t_2+m+1)x^2+(t_1t_2+t_2+mt_1)\right] \end{gathered} \tag{3}$$

So we need t_2 to be a square and

$$x^4-(t_1+t_2+m+1)x^2+(t_1t_2+t_2+mt_1) = (x^2-b_1^2)(x^2-b_2^2).$$

Thus we have 3 conditions

1. $b_1^2 + b_2^2 = t_1 + t_2 + m + 1$
2. $b_1^2b_2^2 = t_1t_2 + t_2 + mt_1$
3. t_2 is a square.

Since $m > 1$ and $t_1 \neq t_2$, we have $t_1t_2 + t_2 + mt_1 > 0$ and so $b_1 \neq 0$.

Let $t_2 = b_1^2 - k_1, k_1 \in \mathbb{Z}$. From the first condition we get $t_1 = b_2^2 + k_1 - m - 1$. By substituting these expressions for t_1 and t_2 into the second condition we get

$$b_2^2(k_1-m) = b_1^2(k_1-m)-(k_1-m)^2-m. \tag{4}$$

We can divide by $k_1 - m$ in Equation (4) (we know $k_1 - m \neq 0$ since $k_1 - m = 0 \Rightarrow m = 0$) and then we get

$$b_2^2 = b_1^2+m-k_1-\frac{m}{k_1-m}. \tag{5}$$

In order for $\frac{m}{k_1-m}$ to be an integer we must have $(k_1 - m)|m$. If we let $a_1 = k_1 - m$, $m = a_1k_2$ for some $k_2 \in \mathbb{Z}\backslash\{0\}$. Now since $m > 1$, a_1 and k_2 must have the same sign. By making the substitutions into Equation (5) we get

$$b_2^2 = b_1^2-a_1-k_2. \tag{6}$$

Now by substituting in the expressions for k_1 and m we get

$$t_1 = b_2^2 + a_1 - 1 \tag{7}$$
$$t_2 = b_1^2 - a_1k_2 - a_1 \tag{8}$$

Note b_2^2 and t_2 are both squares hence there exist $a_2, a_3 \in \mathbb{Z}$ such that

$$b_2^2 = (b_1 - a_2)^2 \tag{9}$$
$$t_2 = (b_1 - a_3)^2 \tag{10}$$

By comparing Equations (6) and (8) with Equations (9) and (10) we get

$$a_1 + k_2 = 2b_1a_2 - a_2^2 \tag{11}$$
$$a_1(k_2 + 1) = 2b_1a_3 - a_3^2. \tag{12}$$

Solving Equation (11) for k_2 and substituting into Equation (12) we get

$$a_3^2 - a_1a_2^2 - a_1^2 + a_1 = 2b_1(a_3 - a_1a_2). \tag{13}$$

We will look at two cases, $a_3 = a_1a_2$ and $a_3 \neq a_1a_2$, separately.

Case 1. If $a_3 = a_1a_2$ then from Equation (13) we get

$$\begin{aligned} 0 &= a_3^2 - a_1a_2^2 - a_1^2 + a_1 \\ &= a_2^2a_1^2 - a_1a_2^2 - a_1^2 + a_1 \\ &= a_1(a_1 - 1)(a_2 - 1)(a_2 + 1). \end{aligned}$$

So $a_1 = 0$ or 1 or $a_2 = \pm 1$. Now $a_1 \neq 0$ since $m > 0$. If $a_1 = 1$ then $a_3 = a_2a_1 = a_2$, and so $b_2^2 = (b_1 - a_2)^2 = (b_1 - a_3)^2 = t_2$ and $t_1 = b_2^2 + a_1 - 1 = b_2^2 = t_2$. But we assumed $t_1 \neq t_2$. So $a_1 \neq 1$.

If $a_2 = -1$ then $a_3 = -a_1$, $m = -a_1(2b_1 + 1 + a_1)$, $t_1 = b_1^2 + 2b_1 + a_1$ and $t_2 = (b_1 + a_1)^2$. If $a_2 = 1$ then the result is identical to the case with $a_2 = -1$. Now we see that $T(1, t_1) \bullet T(m, t_2)$ is integral with $x^4 - (t_1 + t_2 + m + 1)x^2 + (t_1t_2 + t_2 + mt_1) = (x^2 - b_1^2)(x^2 - (b_1 + 1)^2)$.

It remains to ensure that $m > 0$ and $t_1, t_2 \geq 0$, hence we have two conditions:

1. $m = -a_1(2b_1 + 1 + a_1) > 0$
2. $t_1 = b_1^2 + 2b_1 + a_1 \geq 0$.

For the first condition $-a_1(2b_1 + 1 + a_1) > 0$ hence b_1 and a_1 have different sign. Suppose $a_1 < 0 < b_1$ then we require $-a_1(2b_1 + 1 + a_1) > 0$ and $b_1^2 + 2b_1 + a_1 \geq 0$. Since $a_1 < 0$ we see from the first condition that $2b_1 + 1 + a_1 > 0$ and hence $-a_1 < 2b_1 + 1$, and from the second condition $-a_1 \leq b_1^2 + 2b_1$. Since a_1 and b_1 are integers it follows that the condition $-a_1 < b_1^2 + 2b_1 + 1 = (b_1 + 1)^2$ and $-a_1 \leq b_1^2 + 2b_1$ are equivalent thus $0 < -a_1 < min\{(b_1 + 1)^2, 2b_1 + 1\}$. Suppose $b_1 < 0 < a_1$, then we require $2b_1 + 1 + a_1 < 0$

and $b_1^2 + 2b_1 + a_1 \geq 0$ hence $-b_1^2 - 2b_1 \leq a_1 < -2b_1 - 1$. If $b_1 = -1$ then we obtain $1 < a_1 < 1$ which has no solution; if $b_1 \leq -2$ then $-b_1^2 - 2b_1 \leq 0$ and we have $a_1 > 0$ hence $0 < a_1 < -2b_1 - 1$. This completes Case 1.

Case 2. If $a_3 \neq a_1 a_2$ then from Equation (13) we get

$$b_1 = \frac{a_1 a_2 + a_3}{2} + \frac{a_1(a_1 - 1)(a_2 + 1)(a_2 - 1)}{2(a_3 - a_1 a_2)} \tag{14}$$

with

$$\begin{aligned} m &= a_1 k_2 \\ &= a_1(2b_1 a_2 - a_2^2 - a_1) \end{aligned}$$

and

$$t_2 = (b_1 - a_3)^2$$

and finally

$$\begin{aligned} t_1 &= b_2^2 + k_1 - m - 1 \\ &= (b_1 - a_2)^2 + a_1 - 1 \end{aligned}$$

So whenever $a_1, a_2, a_3 \in \mathbb{Z}$ such that $b_1 \in \mathbb{Z}, m > 0$ and $t_1, t_2 \geq 0$ then Equation (3) will have all integer solutions. □

Example 3.2. *We consider Case 1 of Theorem 3.1. Table 1 lists some small examples where b_1 is the b of the Theorem.*

Table 1: Examples of Case 1 in Theorem 3.1

a	b_1	b_2	m	t_1	t_2
2	-6	5	18	26	16
3	-6	5	24	27	9
4	-6	5	28	28	4
5	-6	5	30	29	1
7	-6	5	28	31	1
8	-6	5	24	32	4
9	-6	5	18	33	9
10	-6	5	10	34	16
2	-5	4	14	17	9
3	-5	4	18	18	4
4	-5	4	20	19	1

a	b_1	b_2	m	t_1	t_2
6	-5	4	18	21	1
7	-5	4	14	22	4
8	-5	4	8	23	9
2	-4	3	10	10	4
3	-4	3	12	11	1
5	-4	3	10	13	1
6	-4	3	6	14	4
2	-3	2	6	5	1
4	-3	2	4	7	1

For example, if we let $b = -10$, then we can choose a such that $0 < a < 19$, so for example let $a = 17$. Then $t_1 = (-10 + 17)^2 = 49$, $t_2 = (-10)^2 + 2 \times (-10) + 17 = 97$, $m = -17(2 \times (-10) + 1 + 17) = 34$. So $T(34, 49) \bullet T(1, 97)$ is an integral tree.

Example 3.3. *Let's look at Case 2 in Theorem 3.1. There are many values of the parameters a_1, a_2, a_3 for which b is an integer, thus producing integral trees. Table 2 lists some small examples. The trees of smallest order are $T(1, 5) \bullet T(6, 1)$ and $T(1, 7) \bullet T(4, 1)$. In the Corollary below, we note one particularly simple infinite family.*

Table 2: Examples of Case 2 in Theorem 3.1

a_1	a_2	a_3	b_1	b_2	m	t_1	t_2
-7	-1	1	4	5	14	17	19
-7	-1	3	5	6	28	28	4
-6	-1	2	4	5	18	18	4
-5	-7	-5	-3	4	10	10	4
-3	-1	7	5	6	24	32	4
-2	-1	6	4	5	14	22	4
-1	-7	-5	-3	4	6	14	4
-1	-5	7	18	23	204	527	121
-1	-3	5	8	11	56	119	9
-1	-1	7	4	5	8	23	9

In the case where $a_2 = 1$ and a_1 and a_3 have the same parity, b will always be an integer. So we have the following infinite family of integral trees, without any further restriction on the parameters, except that $a_1 \neq a_3$:

Corollary 3.4. *Suppose a_1 and a_3 have the same parity and $a_1 \neq a_3$. Then $T(1, t_1) \bullet T(m, t_2)$ is an integral tree if $t_2 = (a_1 - a_3)^2/4$, $t_1 = (a_1 + a_3 - 2)^2/4 + a_1 - 1$, $m = a_1(a_3 - 1)$.*

Proof. This is the case when $a_2 = 1$ in Case 2. □

4. Trees of type $T(m_1, t_1) \bullet T(m_2, t_2)$

In this section we will examine integral trees of the more general form $T(m_1, t_1) \bullet T(m_2, t_2)$. Theorems 2.2 and 3.1 dealt with the cases where $t_1 = t_2$ and $m_1 = 1$ so we will now assume that $m_1, m_2 > 1$ and $t_1 \neq t_2$.

Theorem 4.1. *Suppose $T(m_1, t_1) \bullet T(m_2, t_2)$ is integral and $m_1, m_2 > 1$ and $t_1 \neq t_2$, then there exist integer parameters a_1, a_1, a_3, a_4, a_5 such that letting*

$$
\begin{aligned}
c_1 &= \frac{a_1a_3a_5(a_3 + a_5) + a_2a_4a_5(a_4 - 2a_3 - a_5)}{(a_1(a_3 + a_5) - a_2a_4)(a_1 - a_2)}, \\
c_2 &= \frac{a_3a_4(a_1 - a_2)(a_3 - a_4) - a_1a_3a_5(a_3 + a_5) + a_1a_4a_5(a_5 - a_4) + 2a_1a_3a_4a_5}{(a_1(a_3 + a_5) - a_2a_4)(a_1 - a_2)}, \\
b_1 &= \frac{a_1(a_3 + a_5)^2 - a_2a_4^2}{2[a_1(a_3 + a_5) - a_2a_4]}
\end{aligned}
$$

then $m_1 = a_1c_2$, $m_2 = a_2c_1$, $t_1 = (b_1 - a_4)^2$, $t_2 = (b_1 - a_3 - a_5)^2$ whenever $b_1 \in \mathbb{Z}\backslash\{0\}$, $c_1, c_2 \in \mathbb{Z}$, and a_1, c_2 and a_2, c_1 have the same sign.

Proof. From Corollary 2.4 the characteristic equation of $T(m_1, t_1) \bullet T(m_2, t_2)$ is

$$
\begin{aligned}
&x^{1-m_1-m_2+m_1t_1+m_2t_2}(x^2 - t_1)^{m_1-1}(x^2 - t_2)^{m_2-1} \times \\
&\times \left[x^4 - (t_1 + t_2 + m_1 + m_2)x^2 + (t_1t_2 + m_2t_1 + m_1t_2)\right].
\end{aligned} \tag{15}
$$

So we need both t_1 and t_2 to be squares and

$$x^4 - (t_1 + t_2 + m_1 + m_2)x^2 + (t_1t_2 + m_2t_1 + m_1t_2) = (x^2 - b_1^2)(x^2 - b_2^2).$$

Thus we have 3 conditions:

1. $b_1^2 + b_2^2 = t_1 + t_2 + m_1 + m_2$
2. $b_1^2 b_2^2 = t_1t_2 + m_2t_1 + m_1t_2$
3. t_1, t_2 are both squares.

Let $t_1 = b_1^2 - k_1, k_1 \in \mathbb{Z}$, then from the first condition we have $t_2 = b_2^2 + k_1 - m_1 - m_2$. Substituting these into the second condition gives us

$$b_2^2(k_1 - m_1) = b_1^2(k_1 - m_1) - (k_1 - m_1)^2 - m_1m_2. \tag{16}$$

Note that $k_1 \neq m_1$ since otherwise Equation (16) becomes $m_1m_2 = 0$, contradicting $m_1, m_2 > 0$. So we may divide by $k_1 - m_1$ in Equation (16) which then becomes

$$b_2^2 = b_1^2 + m_1 - k_1 - \frac{m_1m_2}{k_1 - m_1} \tag{17}$$

Now since $(k_1 - m_1)|m_1m_2$ we must have $k_1 - m_1 = a_1c_1$ for some $a_1, c_1 \in \mathbb{Z}\backslash\{0\}$ where $m_1 = a_1c_2$ and $m_2 = a_2c_1$ for some $a_2, c_2 \in \mathbb{Z}\backslash\{0\}$. And since $m_1, m_2 > 1$ a_1 and c_2 have the same sign and a_2 and c_1 have the same sign. By making the necessary substitutions into Equation (17) we get

$$b_2^2 = b_1^2 - a_1c_1 - a_2c_2 \tag{18}$$

Also by substituting in the expressions for k_1, m_1, m_2 we get

$$t_1 = b_1^2 - a_1c_2 - a_1c_1 \tag{19}$$
$$t_2 = b_2^2 + a_1c_1 - a_2c_1 \tag{20}$$

We note that b_2^2, t_1 and t_2 are all squares, hence $\exists\ a_3, a_4, a_5 \in \mathbb{Z}$ such that

$$b_2^2 = (b_1 - a_3)^2 \tag{21}$$
$$t_1 = (b_1 - a_4)^2 \tag{22}$$
$$t_2 = (b_2 - a_5)^2 \tag{23}$$

By comparing Equations (18) to (20) with Equations (21) to (23) we get

$$a_1c_1 + a_2c_2 = a_3(2b_1 - a_3) \tag{24}$$
$$a_1(c_2 + c_1) = a_4(2b_1 - a_4) \tag{25}$$
$$c_1(a_1 - a_2) = a_5(a_5 - 2b_2) \tag{26}$$

Claim. $a_1 \neq a_2$

Proof. For contradiction, suppose $a_1 = a_2$. Then

$$\begin{aligned} t_1 &= (b_1 - a_4)^2 \ ; \quad \text{from Equation (22)} \\ &= b_1^2 - 2a_4b_1 + a_4^2 \\ &= b_1^2 - a_1c_2 - a_1c_1 \ ; \ \text{from Equation (25)} \\ &= b_1^2 - a_2c_2 - a_1c_1 \ ; \quad \text{since } a_1 = a_2 \\ &= b_1^2 - 2a_3b_1 + a_3^2 \ ; \quad \text{from Equation (24)} \\ &= (b_1 - a_3)^2 \\ &= b_2^2 \ ; \quad \text{from Equation (21)} \end{aligned}$$

Now from Equation (26) we note that $a_5 = 0$ or $a_5 = 2b_2$ and for both cases Equation (23) becomes $t_2 = b_2^2$, which contradicts $t_1 \neq t_2$. Hence $a_1 \neq a_2$. □

From Equation (21) we note that $b_2 = b_1 - a_3$ or $b_2 = a_3 - b_1$. Suppose, $b_2 = b_1 - a_3$ then because of Claim 1, we know we can rearrange Equation (26) to get

$$c_1 = \frac{a_5(a_5 + 2a_3 - 2b_1)}{a_1 - a_2}. \tag{27}$$

By rearranging Equation (24) we get

$$c_2 = \frac{2a_3b_1 - a_3^2 - a_1c_1}{a_2} \tag{28}$$

$$= \frac{a_3(2b_1 - a_3)}{a_2} - \frac{a_1a_5(a_5 - 2b_1 + 2a_3)}{(a_1 - a_2)a_2} \tag{29}$$

By substituting Equations (27) and (29) into Equation (25) and rearranging terms we get

$$2b_1[a_1(a_3 + a_5) - a_2a_4] = [a_1(a_3 + a_5)^2 - a_2a_4^2] \tag{30}$$

Claim. $[a_1(a_3 + a_5) - a_4a_2] \neq 0$

Proof. For contradiction, suppose $[a_1(a_3 + a_5) - a_4a_2] = 0$. Then from Equation (30) $[a_1(a_3 + a_5)^2 - a_4^2a_2] = 0$ and rearranging terms we get

$$a_4a_2 = a_1(a_3 + a_5) \tag{31}$$

$$a_4^2a_2 = a_1(a_3 + a_5)^2. \tag{32}$$

if $a_4 = 0$ then $t_1 = b_1^2$ and $a_1(a_3 + a_5) = 0$ which implies $a_5 = -a_3$ (since $a_1 \neq 0$ by definition). Hence $t_2 = (b_2 + a_3)^2 = b_1^2$ making $t_1 = t_2$. This is a contradiction hence $a_4 \neq 0$. so

$$a_2 = \frac{a_1(a_3 + a_5)}{a_4} \tag{33}$$

By substituting into Equation (32) we obtain

$$a_4a_1(a_3 + a_5) = a_1(a_3 + a_5)^2 \tag{34}$$

now $a_1 \neq 0$ by definition, and if $a_3 = -a_5$ then, Equation (33) becomes $a_2 = 0$ which contradicts the definition of a_2. By dividing by $a_1(a_3 + a_5)$ we obtain $a_4 = a_3 + a_5$ and Equation (33) becomes $a_1 = a_2$ which contradicts $a_1 \neq a_2$. Hence $[a_1(a_3 + a_5) - a_2a_4] \neq 0$. □

Finally, by rearranging Equation (30) and using $[a_1(a_3 + a_5) - a_2a_4] \neq 0$ we get

$$b_1 = \frac{a_1(a_3 + a_5)^2 - a_2a_4^2}{2[a_1(a_3 + a_5) - a_2a_4]} \tag{35}$$

Substituting Equation (35) into Equations (27) and (29) gives us

$$c_1 = \frac{a_1a_3a_5(a_3 + a_5) + a_2a_4a_5(a_4 - 2a_3 - a_5)}{(a_1(a_3 + a_5) - a_2a_4)(a_1 - a_2)}$$

and

$$c_2 = \frac{a_3a_4(a_1-a_2)(a_3-a_4) - a_1a_3a_5(a_3+a_5) + a_1a_4a_5(a_5-a_4) + 2a_1a_3a_4a_5}{(a_1(a_3+a_5) - a_2a_4)(a_1-a_2)}.$$

If we had supposed $b_2 = a_3 - b_1$, we would have been achieved the same result, except that a_5 would be $-a_5$. However, since $a_5 \in \mathbb{Z}$ the outcome is the same. □

Example 4.2. *Theorem 4.1 is not as cumbersome to apply as it looks at first glance. For example, let* $a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 6, a_5 = 4$. *Then* $b_1 = 47/14, c_2 = 10/7, c_1 = 20/7$, *and* $t_1 = 1369/196, t_2 = 529/196, m_1 = 10/7, m_2 = 40/7$. *Since these values are not integers, we use Corollary 2.5 to scale up the rational values of the parameters* t_1, t_2, m_1, m_2. *To do so we must multiply by a suitable square, in this case 196, and obtain* $t_1 = 1369, t_2 = 529, m_1 = 280, m_2 = 1120$. *So* $T(m_1, t_1) \bullet T(m_2, t_2)$ *is integral.*

Table 3 shows some examples Theorem 4.1 provides for small values of t_1 *and* t_2 *(keeping in mind that they must be squares). These (among many others) were produced by choosing* $a_1,\ a_2 \in [-6, 6]\backslash\{0\},\ \ a_3,\ a_4,\ a_5 \in [-5, 5]$. *The column labeled* d *refers to the multiplier used to make the* m_i *and* t_i *integers.*

Table 3: Examples from Theorem 4.1

a_1	a_2	a_3	a_4	a_5	d	m_1	m_2	t_1	t_2
-3	5	-5	3	2	1	42	25	9	81
-6	2	-3	5	2	1	72	9	4	64
-4	6	-3	2	1	1	22	9	9	49
-3	-4	-5	-4	-1	1	39	44	49	25
-3	-6	-5	4	1	1	39	14	4	100
-1	-4	-4	-5	2	9	11	16	49	4
-4	-3	-4	-5	1	4	64	72	9	49
-6	2	-5	1	2	16	36	36	121	25
-4	-1	-1	-2	-4	9	28	8	4	49
-5	-3	-3	-5	-4	16	40	144	9	121
-2	6	-4	2	1	4	54	42	9	169
-1	-3	-2	-1	-2	4	20	84	121	25
-4	6	-1	2	5	49	126	21	9	289
-2	6	-3	3	4	25	105	360	64	4

The example suggests how to obtain a family of trees in which the m_i and t_i are already integers. The idea is to incorporate a suitable square factor into the expressions. We obtain the following corollary 4.3.

Corollary 4.3. *Suppose* $a_1, \cdots, a_5$ *are integer parameters with* $a_1, a_2 \neq 0$.
Let $\alpha = a_1a_3^2 + 2a_1a_3a_5 + a_1a_5^2 - a_2a_4^2$,

$\beta = 2(a_1(a_3+a_5) - a_2a_4)$,
$\gamma = a_5(a_1a_3a_5 - a_2a_4a_5 + a_1a_3^2 + a_2a_4^2 - 2a_2a_3a_4)$,
$\delta = (a_1 - a_2)(a_1(a_3+a_5) - a_2a_4)$,
$\lambda = -a_1a_3^2a_5 - a_1a_3a_5^2 - a_1a_3a_4^2 + a_2a_3a_4^2 + a_1a_4a_3^2 - a_2a_3^2a_4 + a_1a_4a_5^2 - a_1a_4^2a_5 + 2a_1a_3a_4a_5$.
If $m_1 = a_1\lambda\delta\beta^4$, $m_2 = \gamma a_2\delta\beta^4$, $t_1 = (\alpha - a_4\beta)^2\delta^2\beta^2$, $t_2 = (\alpha - a_3\beta - a_5\beta)^2\delta^2\beta^2$ *such that* $m_1, m_2 > 0$, $t_1, t_2 \geq 0$, *then* $T(m_1,t_1) \bullet T(m_2,t_2)$ *is an integral tree.*

Proof. It is easy to see that t_1 and t_2 are squares. Direct substitution shows that the characteristic polynomial factors as required:

$$\begin{aligned} & x^4 - (t_1 + t_2 + m_2 + m_1)x^2 + (t_1t_2 + m_2t_1 + m_1t_2) \\ = \quad & (x^2 - (\alpha\delta\beta)^2)(x^2 - ((\alpha - \beta a_3)\delta\beta)^2) \end{aligned}$$

□

5. Trees of the form $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$

In the previous section, we gave a complete characterization of trees of the form $T(m_1,t_1) \bullet T(m_2,t_2)$. Of course we would like to do so for trees $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$ where $n > 2$, but we are not yet able to. However we are able to provide infinite families of these trees for each $n > 1$. The t_i can be arbitrary distinct squares, and the m_i are given by particularly elegant formulas. As we did in the previous Section, we can scale up the rational m_i using Corollary 2.5 if they are not integer.

Theorem 5.1. *Let* $\{a_1, \cdots, a_n\}$ *and* $\{b_1, \cdots, b_n\}$ *be two sets of distinct non-negative integer parameters and let* $t_i = a_i^2$ *and* $m_i = \frac{\prod_{j=1}^n (b_j^2 - t_i)}{\prod_{j=1, j\neq i}^n (t_j - t_i)}$. *Then if all* $m_i > 0$, $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$ *yields an integral tree by suitable scaling.*

Proof. From Corollary 2.4, the characteristic equation for $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$ is

$$x^{1+\sum_{i=1}^n m_it_i - m_i} \left[\prod_{i=1}^n \left(x^2 - t_i\right)^{m_i} - \sum_{i=1}^n m_i \left(x^2 - t_i\right)^{m_i - 1} \prod_{j=1, j\neq i}^n \left(x^2 - t_j\right)^{m_j} \right]$$

Now $t_i = a_i^2$, so the characteristic equation for $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$ can be simplified to

$$x^{1+\sum_{i=1}^n m_it_i - m_i} \prod_{i=1}^n \left(x^2 - a_i^2\right)^{m_i - 1} \left[\prod_{i=1}^n \left(x^2 - a_i^2\right) - \sum_{i=1}^n m_i \prod_{j=1, j\neq i}^n \left(x^2 - a_j^2\right) \right]$$

So $T(m_1,t_1) \bullet \cdots \bullet T(m_n,t_n)$ is integral if

$$\left[\prod_{i=1}^n \left(x^2 - a_i^2\right) - \sum_{i=1}^n m_i \prod_{j=1, j\neq i}^n \left(x^2 - a_j^2\right) \right] \quad = \quad 0$$

has only integer roots.

Now

$$m_i = \frac{\prod_{j=1}^{n}\left(b_j^2 - t_i\right)}{\prod_{j=1,j\neq i}^{n}(t_j - t_i)}$$

so $T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$ is integral if

$$\prod_{i=1}^{n}(x^2 - a_i^2) - \sum_{i=1}^{n}\prod_{j=1}^{n}(b_j^2 - a_i^2)\prod_{j=1,j\neq i}^{n}\left(\frac{x^2 - a_j^2}{a_j^2 - a_i^2}\right) = 0 \tag{36}$$

has only integer roots.

Claim.

$$\prod_{i=1}^{n}(x^2 - a_i^2) - \sum_{i=1}^{n}\prod_{j=1}^{n}(b_j^2 - a_i^2)\prod_{j=1,j\neq i}^{n}\left(\frac{x^2 - a_j^2}{a_j^2 - a_i^2}\right) = \prod_{j=1}^{n}(x^2 - b_j^2)$$

Proof. Proof by contradiction. Let

$$\begin{aligned} q(x) &= \prod_{i=1}^{n}(x^2 - a_i^2) - \sum_{i=1}^{n}\left[\prod_{j=1}^{n}(b_j^2 - a_i^2)\prod_{j=1,j\neq i}^{n}\left(\frac{x^2 - a_j^2}{a_j^2 - a_i^2}\right)\right] \\ &\quad - \prod_{j=1}^{n}(x^2 - b_j^2) \end{aligned}$$

If $q(x) = 0\ \forall x$ then trivially the claim is proved. Suppose the $q(x) \neq 0$. By definition $q(x)$ is a polynomial of degree k where $k \leq 2(n-1)$, and since $q(x) \neq 0$, $q(x)$ has at most $2n-2$ roots. Let $1 \leq k \leq n$ then

$$\begin{aligned} q(\pm a_k) &= \prod_{i=1}^{n}((\pm a_k)^2 - a_i^2) - \prod_{j=1}^{n}((\pm a_k)^2 - b_j^2) \\ &\qquad - \sum_{i=1}^{n}\prod_{j=1}^{n}(b_j^2 - a_i^2)\prod_{j=1,j\neq i}^{n}\left(\frac{(\pm a_k)^2 - a_j^2}{a_j^2 - a_i^2}\right) \\ &= -\prod_{j=1}^{n}(b_j^2 - a_k^2)\prod_{j=1,j\neq i}^{n}\left(\frac{a_k^2 - a_j^2}{a_j^2 - a_k^2}\right) - \prod_{j=1}^{n}(a_k^2 - b_j^2) \\ &= (-1)^n\prod_{j=1}^{n}(b_j^2 - a_k^2) - \prod_{j=1}^{n}(a_k^2 - b_j^2) \end{aligned}$$

$$\begin{aligned} &= \prod_{j=1}^{n} -\left(b_j^2 - a_k^2\right) - \prod_{j=1}^{n}\left(a_k^2 - b_j^2\right) \\ &= \prod_{j=1}^{n}\left(a_k^2 - b_j^2\right) - \prod_{j=1}^{n}\left(a_k^2 - b_j^2\right) \\ &= 0 \end{aligned}$$

Note at most only one $a_i = 0$ hence $q(x)$ has at least $2n-1$ roots, contradicting $q(x)$ having at most $2(n-1)$ roots. This proves the claim. □

So

$$\left[\prod_{i=1}^{n}\left(x^2 - a_i^2\right) - \sum_{i=1}^{n} m_i \prod_{j=1, j\neq i}^{n}\left(x^2 - a_j^2\right)\right] = 0$$

has only rational roots, and $T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$ can be scaled up to an integral tree using Corollary 2.5. □

We provide some examples of the integral trees generated by the method of Theorem 5.1. Tables 4 and 5 contains examples where $n = 3$ and $n = 4$ respectively. In Table 4 we include a column headed d indicating that the scale factor d^2 was used when applying Corollary 2.5.

Table 4: Examples from Theorem 5.1 with $n = 3$

a_1	a_2	a_3	b_1	b_2	b_3	d	$T(m_1,t_1) \bullet T(m_2,t_2) \bullet T(m_3,t_3)$
5	3	1	2	4	7	4	$T(189,400) \bullet T(175,144) \bullet T(180,16)$
5	3	1	2	4	8	16	$T(4914,6400) \bullet T(3850,2304) \bullet T(3780,256)$
7	3	1	2	4	8	16	$T(2970,12544) \bullet T(1540,2304) \bullet T(1890,256)$
6	4	1	2	5	8	5	$T(352,900) \bullet T(432,400) \bullet T(216,25)$
7	3	1	2	5	8	4	$T(135,784) \bullet T(220,144) \bullet T(189,16)$
6	4	1	3	5	8	5	$T(297,900) \bullet T(252,400) \bullet T(576,25)$
7	5	1	3	6	8	12	$T(975,7056) \bullet T(1716,3600) \bullet T(2205,144)$
7	5	1	2	6	8	16	$T(1950,12544) \bullet T(4004,6400) \bullet T(1470,256)$

Example 5.2. *There is one special case of Theorem 5.1 where the m_i simplify particularly*

Table 5: Examples from Theorem 5.1 with $n = 4$

(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), d, $T(m_1, t_1) \bullet T(m_2, t_2) \bullet T(m_3, t_3) \bullet T(m_4, t_4)$
$(7, 5, 3, 1)$, $(2, 4, 6, 8)$, 32 $T(6435, 50176) \bullet T(9009, 25600) \bullet T(10395, 9216) \bullet T(11025, 1024)$
$(7, 5, 3, 1)$, $(2, 4, 6, 9)$, 16 $T(3432, 12544) \bullet T(3234, 6400) \bullet T(3402, 2304) \bullet T(3500, 256)$
$(7, 5, 3, 1)$, $(2, 4, 6, 10)$, 32 $T(21879, 50176) \bullet T(17325, 25600) \bullet T(17199, 9216) \bullet T(17325, 1024)$
$(9, 5, 3, 1)$, $(2, 4, 7, 10)$, 481 $T(21736, 186624) \bullet T(36450, 57600) \bullet T(31850, 20736) \bullet T(32076, 2304)$
$(8, 6, 4, 1)$, $(2, 5, 7, 10)$, 70 $T(73125, 313600) \bullet T(73216, 176400) \bullet T(101871, 78400) \bullet T(50688, 4900)$
$(9, 7, 5, 1)$, $(2, 6, 8, 10)$, 64 $T(31977, 331776) \bullet T(49725, 200704) \bullet T(85800, 102400) \bullet T(29106, 4096)$

well. Let $a_i = 2i - 1$ and $b_i = 2i$, with $t_i = a_i^2$. Then trivially $t_i = (2i-1)^2$, and

$$\begin{aligned} m_i &= \frac{\prod_{j=1}^{n}(2j)^2 - (2i-1)^2}{\prod_{j=1, j\neq i}^{n}((2j-1)^2 - (2i-1)^2)} \\ &= \frac{\prod_{j=1}^{n}(2j-2i+1)(2j+2i-1)}{\prod_{j=1, j\neq i}^{n} 4(j-i)(j+i-1)} \\ &= (4i-1)\prod_{j=1, j\neq i}^{n}\left(\frac{(2j-2i+1)(2j+2i-1)}{4(j-i)(j+i-1)}\right). \end{aligned}$$

Now since $i \neq j$ we have $m_i > 0 \; \forall \; i$. Hence $T(d_n^2 m_1, d_n^2 t_1) \bullet \cdots \bullet T(d_n^2 m_n, d_n^2 t_n)$ can be scaled up to an integral tree for all n. Choosing $d_n = 2^{2(n-1)}$ was found to be sufficient to make m_i an integer for $n \leq 1000$. With d_n so defined, Table 6 lists the trees we get.

Table 6: The integral trees from example 5.2

n	$T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$
1	$T(3, 1)$
2	$T(90, 16) \bullet T(70, 144)$
3	$T(2100, 256) \bullet T(1890, 2304) \bullet T(1386, 6400)$
4	$T(44100, 4096) \bullet T(41580, 36864) \bullet T(36036, 102400) \bullet T(25740, 200704)$

We believe that Theorem 5.1 characterizes all integral trees of diameter 4 where all $m_i > 1$. We state this formally as

Conjecture 5.3. *If $T = T(m_1, t_1) \bullet \cdots \bullet T(m_n, t_n)$ is an integral tree with $m_i > 1 \forall\ i$ and all t_i distinct, then $\exists\ a_i, b_i \in \mathbb{N} \cup \{0\}$ such that T is obtained by scaling from $t_i = a_i^2$, $m_i = \frac{\prod_{j=1}^{n} b_j^2 - t_i}{\prod_{j=1, j\neq i}^{n}(t_j - t_i)}$.*

We note that requiring the t_i to be distinct in Theorem 5.1 is no restriction, since $T(m_i, t_i) \bullet T(m_j, t_i)$ would simply collapse to $T(m_i + m_j, t_i)$.

It is important to note that Theorem 5.1 does not apply to the case where one or more $m_i = 1$. Consequently work remains to be done to discover expressions for the t_i and m_i which will yield all of these trees. While the formulas of Theorem 5.1 are still valid, integral trees can also be produced in which the t_i are not squares.

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