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Signed total (k, k)-domatic number of a graph

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Abstract

Let G be a finite and simple graph with vertex set $V(G)$, and let $f : V(G) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of v, then f is a signed total k-dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of signed total k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$, is called a signed total (k, k)-dominating family (of functions) on G. The maximum number of functions in a signed total (k, k)-dominating family on G is the signed total (k, k)-domatic number on G, denoted by $d_{st}^k(G)$.

In this paper we initiate the study of the signed total (k, k)-domatic number, and we present different bounds on $d_{st}^k(G)$. Some of our results are extensions of known properties of the signed total domatic number $d_{st}(G) = d_{st}^1(G)$, given by Henning in 2006.

Keywords: Signed total (k, k)-domatic number, signed total k-dominating function, signed total k-domination number.

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1. Terminology and introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants (see, for example, Haynes, Hedetniemi and Slater [1]). In this paper we define the *signed total (k, k)-domatic number* in an analogous way as Henning [3] have introduced the signed total domatic number.

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $n = n(G)$. If $v \in V(G)$, then $N(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent to v. The *closed neighborhood* $N[v]$ of a vertex v consists of the vertex set $N(v) \cup \{v\}$. The number $d(v) = |N(v)|$ is the *degree* of the vertex v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. The *complement* of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. A *fan* and a *wheel* is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle, respectively. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

If $k \geq 1$ is an integer, then the *signed total k-dominating function* is defined by Wang [4] as a two-valued function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed total k-dominating functions f on G, is called the *signed total k-domination number* of G, denoted by $\gamma_{st}^k(G)$. A $\gamma_{st}^k(G)$-*function* is a signed total k-dominating function on G with weight $\gamma_{st}^k(G)$. As the assumption $\delta(G) \geq k$ is necessary, we always assume that when we discuss $\gamma_{st}^k(G)$, all graphs involved satisfy $\delta(G) \geq k$ and thus $n(G) \geq k + 1$. The special case $k = 1$ was defined by Zelinka in [5]. Further results on $\gamma_{st}(G) = \gamma_{st}^1(G)$ can be found in the paper [2] of Henning.

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total k-dominating functions on G such that $\sum_{i=1}^d f_i(x) \leq k$ for each $x \in V(G)$, is called a *signed total (k, k)-dominating family* on G. The maximum number of functions in a signed total (k, k)-dominating family on G is the *signed total (k, k)-domatic number* of G, denoted by $d_{st}^k(G)$.

First we study basic properties of $d_{st}^k(G)$. Some of them are extensions of well-known results on the signed total domatic number $d_{st}(G) = d_{st}^1(G)$, given by Henning in [3]. Using these results, we determine the signed total (k, k)-domatic numbers of fans, wheels and grids.

In this paper we make use of the following observations.

Observation 1.1. *If G is a graph of order n, $\delta(G) \geq k$ and $\gamma_{st}^k(G) = n$, then $d_{st}^k(G) = 1$.*

Observation 1.2. *If G is a graph of order $n \geq 3$ and $k = n - 1$ or $k = n - 2$, then $\gamma_{st}^k(G) = n$ and thus $d_{st}^k(G) = 1$.*

Observation 1.3. *Let G be a graph of order n with $\delta(G) \geq k$. Then $\gamma_{st}^k(G) = n$ if and only if for each $v \in V(G)$, there exists a vertex $u \in N(v)$ such that $k \leq d(u) \leq k + 1$.*

Proof. Assume that $N(v)$ contains a vertex of degree at most $k + 1$ for every vertex $v \in V(G)$, and let f be a signed total k-dominating function on G. If $x \in N(v)$ such that $d(x) \leq k + 1$, then it follows that $f(v) = 1$. Hence $f(v) = 1$ for each $v \in V(G)$ and thus $\gamma_{st}^k(G) = n$.

Conversely, assume that $\gamma_{st}^k(G) = n$. Suppose to the contrary that G contains a vertex w such $d(u) \geq k + 2$ for each $u \in N(w)$. Then the function $f : V(G) \to \{-1, 1\}$ such that $f(w) = -1$ and $f(x) = 1$ for each vertex $x \in V(G) \setminus \{w\}$ is a signed total k-dominating functions on G. This yields to the contradiction $\gamma_{st}^k(G) \leq n-2$, and the proof is complete. □

2. Basic properties of the signed total (k,k)-domatic number

In this section we present basic properties of $d_{st}^k(G)$ and sharp bounds on the signed total (k,k)-domatic number of a graph.

Proposition 2.1. *The signed total (k,k)-domatic number $d_{st}^k(G)$ is well-defined for each graph G with $\delta(G) \geq k$.*

Proof. Since $\delta(G) \geq k$, the function $f : V(G) \to \{-1, 1\}$ with $f(v) = 1$ for each $v \in V(G)$ is a signed total k-dominating function on G. Thus the family $\{f\}$ is a signed total (k,k)-dominating family on G. Therefore the set of signed total k-dominating functions on G is non-empty and there exists the maximum of their cardinalities, which is the signed total (k,k)-domatic number of G. □

Proposition 2.2. *If G is a graph of order n, then*

$$\gamma_{st}^k(G) \cdot d_{st}^k(G) \leq k \cdot n.$$

Moreover, if $\gamma_{st}^k(G) \cdot d_{st}^k(G) = k \cdot n$, then for each $d = d_{st}^k$-family $\{f_1, f_2, \ldots, f_d\}$ on G each function f_i is a $\gamma_{st}^k(G)$-function and $\sum_{i=1}^d f_i(x) = k$ for all $x \in V(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed total (k,k)-dominating family on G such that $d = d_{st}^k(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{st}^k(G) &= \sum_{i=1}^d \gamma_{st}^k(G) \leq \sum_{i=1}^d \sum_{x \in V(G)} f_i(x) \\ &= \sum_{x \in V(G)} \sum_{i=1}^d f_i(x) \leq \sum_{x \in V(G)} k = k \cdot n. \end{aligned}$$

If $\gamma_{st}^k(G) \cdot d_{st}^k(G) = k \cdot n$, then the two inequalities occurring in the proof become equalities. Hence, for the $d_{st}^k(G)$-family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{x \in V(G)} f_i(x) = \gamma_{st}^k(G)$. Therefore each function f_i is a $\gamma_{st}^k(G)$-function, and $\sum_{i=1}^d f_i(x) = k$ for all x. □

Proposition 2.3. *If $k \geq 1$ is an integer and G a graph of minimum degree $\delta(G) \geq k$, then*

$$d_{st}^k(G) \leq \delta(G).$$

Moreover, if $d_{st}^k(G) = \delta(G)$, then for each function of any signed total (k,k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{st}^k(G)$, and for all vertices v of degree $\delta(G)$, $\sum_{x \in N(v)} f_i(x) = k$ and $\sum_{i=1}^d f_i(x) = k$ for every $x \in N(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (k,k)-dominating family on G such that $d = d_{st}^k(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} d \cdot k &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) \\ &= \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \\ &\leq \sum_{x \in N(v)} k = k \cdot \delta(G), \end{aligned}$$

and this implies the desired upper bound on the signed total (k,k)-domatic number.

If $d_{st}^k(G) = \delta(G)$, then the two inequalities occuring in the proof become equalities, which leads to the two properties given in the statement. □

The special case $k = 1$ in Propositions 2.1, 2.2 and 2.3 can be found in [3]. As an application of Proposition 2.3, we will prove the following Nordhaus-Gaddum type result.

Corollary 2.4. *If $k \geq 1$ is an integer and G a graph of order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, then*

$$d_{st}^k(G) + d_{st}^k(\overline{G}) \leq n - 1.$$

If $d_{st}^k(G) + d_{st}^k(\overline{G}) = n - 1$, then G is regular.

Proof. Since $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, it follows from Proposition 2.3 that

$$\begin{aligned} d_{st}^k(G) + d_{st}^k(\overline{G}) &\leq \delta(G) + \delta(\overline{G}) \\ &= \delta(G) + (n - \Delta(G) - 1) \\ &\leq n - 1, \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{st}^k(G) + d_{st}^k(\overline{G}) \leq n-2$. This completes the proof. □

Theorem 2.5. *If v is a vertex of a graph G such that $d(v)$ is odd and k is even or $d(v)$ is even and k is odd, then*

$$d_{st}^k(G) \leq \frac{k}{k+1} \cdot d(v).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (k,k)-dominating family on G such that $d = d_{st}^k(G)$. Assume first that $d(v)$ is odd and k is even. The definition yields to $\sum_{x \in N(v)} f_i(x) \geq k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{x\in N(v)} f_i(x) \geq k+1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$\begin{aligned} k \cdot d(v) &= \sum_{x\in N(v)} k \geq \sum_{x\in N(v)} \sum_{i=1}^{d} f_i(x) \\ &= \sum_{i=1}^{d} \sum_{x\in N(v)} f_i(x) \\ &\geq \sum_{i=1}^{d} (k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound. Assume next that $d(v)$ is even and k is odd. Note that $\sum_{x\in N(v)} f_i(x) \geq k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{x\in N(v)} f_i(x) \geq k+1$ for each $i \in \{1, 2, \ldots, d\}$. Now the desired bound follows as above, and the proof is complete. □

The next result is an immediate consequence of Theorem 2.5.

Corollary 2.6. *If G is a graph such that $\delta(G)$ is odd and k is even or $\delta(G)$ is even and k is odd, then*

$$d_{st}^{k}(G) \leq \frac{k}{k+1} \cdot \delta(G).$$

The bound is sharp for the cycles when $k = 1$.

As an Application of Corollary 2.6, we will improve the Nordhaus-Gaddum bound in Corollary 2.4 for many cases.

Theorem 2.7. *Let $k \geq 1$ be an integer, and let G be a graph of order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$. If $\Delta(G) - \delta(G) \geq 1$ or k is odd or k is even and $\delta(G)$ is odd or k, $\delta(G)$ and n are even, then*

$$d_{st}^{k}(G) + d_{st}^{k}(\overline{G}) \leq n - 2.$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Corollary 2.4 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1. Assume that k is odd. If $\delta(G)$ is even, then it follows from Proposition 2.3 and Corollary 2.6 that

$$\begin{aligned} d_{st}^{k}(G) + d_{st}^{k}(\overline{G}) &\leq \frac{k}{k+1}\delta(G) + \delta(\overline{G}) \\ &= \frac{k}{k+1}\delta(G) + (n - \delta(G) - 1) \\ &< n - 1, \end{aligned}$$

and we obtain the desired bound. If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Proposition 2.3 and Corollary 2.6, we find that

$$\begin{aligned} d_{st}^k(G) + d_{st}^k(\overline{G}) &\leq \delta(G) + \frac{k}{k+1}\delta(\overline{G}) \\ &= (n - \delta(\overline{G}) - 1) + \frac{k}{k+1}\delta(\overline{G}) \\ &< n - 1, \end{aligned}$$

and this completes the proof of Case 1.

Case 2. Assume that k is even. If $\delta(G)$ is odd, then it follows from Proposition 2.3 and Corollary 2.6 that

$$d_{st}^k(G) + d_{st}^k(\overline{G}) \leq \frac{k}{k+1}\delta(G) + (n - \delta(G) - 1) < n - 1.$$

If $\delta(G)$ is even and n is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Theorem 2.8. *If G is a graph such that k is odd and $d_{st}^k(G)$ is even or k is even and $d_{st}^k(G)$ is odd, then*

$$d_{st}^k(G) \leq \frac{k-1}{k} \cdot \delta(G).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed total (k,k)-dominating family on G such that $d = d_{st}^k(G)$. Assume first that k is odd and d is even. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \leq k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{i=1}^d f_i(x) \leq k-1$ for each $x \in V(G)$. If v is a vertex of minimum degree, then it follows that

$$\begin{aligned} d \cdot k &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) \\ &= \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \\ &\leq \sum_{x \in N(v)} (k-1) = \delta(G)(k-1), \end{aligned}$$

and this yields to the desired bound. Assume second that k is even and d is odd. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \leq k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{i=1}^d f_i(x) \leq k-1$ for each $x \in V(G)$. Now the desired bound follows as above, and the proof is complete. □

According to Proposition 2.1, $d_{st}^k(G)$ is a positive integer. If we suppose in the case $k = 1$ that $d_{st}(G) = d_{st}^1(G)$ is an even integer, then Theorem 2.8 leads to the contradiction $d_{st}(G) \leq 0$. Consequently, we obtain the next known result.

Corollary 2.9. [3] *The signed total domatic number $d_{st}(G)$ is an odd integer.*

Proposition 2.10. *Let $k \geq 2$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. Then $d_{st}^k(G) = 1$ if and only if $N(v)$ contains a vertex of degree at most $k+1$ for every vertex $v \in V(G)$.*

Proof. Assume that $N(v)$ contains a vertex of degree at most $k + 1$ for every vertex $v \in V(G)$. It follows from Observations 1.3 and 1.1 that $\gamma_{st}^k(G) = n$ and thus $d_{st}^k(G) = 1$.

Conversely, assume that $d_{st}^k(G) = 1$. If G contains a vertex w such that $d(x) \geq k + 2$ for each $x \in N(w)$, then the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(G)$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(G) \setminus \{w\}$ are signed total k-dominating functions on G such that $f_1(x) + f_2(x) \leq 2 \leq k$ for each vertex $x \in V(G)$. Thus $\{f_1, f_2\}$ is a signed total (k, k)-dominating family on G, a contradiction to $d_{st}^k(G) = 1$. □

The next result is an immediate consequence of Proposition 2.10.

Corollary 2.11. *Let $k \geq 2$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. Then $d_{st}^k(G) = 1$ if and only if $\gamma_{st}^k(G) = n$.*

Next we present a lower bound on the signed total (k, k)-domatic number.

Proposition 2.12. *Let $k \geq 1$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $k+2$, then $d_{st}^k(G) \geq k$.*

Proof. Let $\{u_1, u_2, \ldots, u_k\} \subset N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $k + 2$ implies that the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_i(u_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{u_i\}$ are signed total k-dominating functions on G for $i \in \{1, 2, \ldots, k\}$. Since $f_1(x) + f_2(x) + \ldots + f_k(x) \leq k$ for each vertex $x \in V(G)$, we observe that $\{f_1, f_2, \ldots, f_k\}$ is a signed total (k, k)-dominating family on G, and Proposition 2.12 is proved. □

Corollary 2.13. *If G is a graph of minimum degree $\delta(G) \geq k+2$, then $d_{st}^k(G) \geq k$.*

Theorem 2.14. *If $k \geq 1$ is an integer and G a graph of order n and minimum degree $\delta(G) \geq k$, then*

$$d_{st}^k(G) + \gamma_{st}^k(G) \leq n + k.$$

Proof. If $\delta(G)=k$, then it follows from Proposition 2.3 that

$$d^k_{st}(G)+\gamma^k_{st}(G)\le \delta(G)+n=n+k.$$

Assume next that $\delta(G)=k+1$. If $\gamma^k_{st}(G)=n$, then $d^k_{st}(G)=1$ and so

$$d^k_{st}(G)+\gamma^k_{st}(G)=n+1\le n+k.$$

In the case that $\gamma^k_{st}(G)\le n-1$, Proposition 2.3 implies that

$$d^k_{st}(G)+\gamma^k_{st}(G)\le \delta(G)+n-1=n+k.$$

Assume now that $\delta(G)\ge k+2$. According to Proposition 2.2, we have

$$d^k_{st}(G)+\gamma^k_{st}(G)\le d^k_{st}(G)+\frac{kn}{d^k_{st}(G)}.$$

In view of Corollary 2.13, $d^k_{st}(G)\ge k$, and Proposition 2.3 implies that $d^k_{st}(G)\le n$. Using these inequalities, and the fact that the function $g(x)=x+(kn)/x$ is decreasing for $k\le x\le\sqrt{kn}$ and increasing for $\sqrt{kn}\le x\le n$, we obtain

$$d^k_{st}(G)+\gamma^k_{st}(G)\le \max\left\{k+\frac{kn}{k},n+\frac{kn}{n}\right\}=n+k.$$

□

Theorem 2.15. *Let $k\ge 1$ be an integer, and let G be a $(k+2)$-regular graph of order n. If $n\not\equiv 0 \pmod{(k+2)}$, then $d^k_{st}(G)=k$.*

Proof. Let f be an arbitrary signed total k-dominating function on G. If we define the sets $P=\{v\in V(G)\,|\,f(v)=1\}$ and $M=\{v\in V(G)\,|\,f(v)=-1\}$, then we firstly show that

$$|P|\ge\left\lceil\frac{n(k+1)}{k+2}\right\rceil. \tag{1}$$

Because of $\sum_{x\in N(y)}f(x)\ge k$ for each vertex $y\in V(G)$, the $(k+2)$-regularity of G implies that each vertex $u\in P$ is adjacent to at most one vertex in M and each vertex $v\in M$ is adjacent to at least $k+1$ vertices in P. Therefore we obtain

$$|P|\ge|M|(k+1)=(n-|P|)(k+1),$$

and this leads to (1) immediately.

Now let $\{f_1,f_2,\dots,f_d\}$ be a signed total (k,k)-dominating family on G with $d=d^k_{st}(G)$. Since $\sum_{i=1}^d f_i(u)\le k$ for every vertex $u\in V(G)$, each of these sums contains at least $\lceil(d-k)/2\rceil$ summands of value -1. Using this and inequality (1), we see that the sum

$$\sum_{x\in V(G)}\sum_{i=1}^{d}f_i(x)=\sum_{i=1}^{d}\sum_{x\in V(G)}f_i(x) \tag{2}$$

contains at least $n\lceil (d-k)/2\rceil$ summands of value -1 and at least $d\lceil n(k+1)/(k+2)\rceil$ summands of value 1. As the sum (2) consists of exactly dn summands, it follows that

$$n\left\lceil \frac{d-k}{2}\right\rceil + d\left\lceil \frac{n(k+1)}{k+2}\right\rceil \leq dn. \tag{3}$$

It follows from the hypothesis $n \not\equiv 0 \pmod{(k+2)}$ that

$$\left\lceil \frac{n(k+1)}{k+2}\right\rceil > \frac{n(k+1)}{k+2},$$

and thus (3) leads to

$$\frac{n(d-k)}{2} + \frac{dn(k+1)}{k+2} < dn.$$

A simple calculation shows that this inequality implies $d < k+2$ and so $d \leq k+1$. If we suppose that $d = k+1$, then we observe that d and k have different parity. Applying Theorem 2.8, we obtain the contradiction

$$k+1 = d \leq \frac{k-1}{k}(k+2) < k+1.$$

Therefore $d \leq k$, and Corollary 2.13 yields to the desired result $d = k$. □

Corollary 2.16. [3] *Let G be a cubic graph of order n. If $d_{st}(G) = 3$, then $n \equiv 0 \pmod 6$.*

On the one hand Theorem 2.15 demonstrates that the bound in Corollary 2.13 is sharp, on the other hand the following example shows that Theorem 2.15 is not valid in general when $n \equiv 0 \pmod{(k+2)}$.

Let $K_{k+2,k+2}$ be the complete bipartite graph with the partite sets $u_1, u_2, \ldots, u_{k+2}$ and $v_1, v_2, \ldots, v_{k+2}$. We define the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_i(u_i) = f_i(v_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{u_i, v_i\}$ and each $i \in \{1, 2, \ldots, k+2\}$. Then we observe that f_i is a signed total k-dominating function on $K_{k+2,k+2}$ for each $i \in \{1, 2, \ldots, k+2\}$ and $\sum_{i=1}^{k+2} f_i(x) = k$ for each vertex $x \in V(K_{k+2,k+2})$. Therefore $\{f_1, f_2, \ldots, f_{k+2}\}$ is a signed total (k,k)-dominating family on $K_{k+2,k+2}$ and thus $d_{st}^k(K_{k+2,k+2}) \geq k+2$. Using Proposition 2.3, we obtain $d_{st}^k(K_{k+2,k+2}) = k+2$.

Because of $d_{st}^k(K_{k+2,k+2}) = k+2$, the equality $d_{st}^k(G) = \delta(G)$ is possible, and thus the bound $d_{st}^k(G) \leq \delta(G)$ in Proposition 2.3 is sharp.

3. Signed total (k,k)-domatic number of fans, wheels and grids

Using some of the results of Section 2, we now determine the signed total (k,k)-domatic numbers of fans, wheels and grids.

Proposition 3.1. *If G is a fan of order $n \geq 3$, then $d_{st}^2(G) = d_{st}(G) = 1$.*

Proof. Since $N(v)$ contains a vertex of degree at most 3 for every vertex $v \in V(G)$, it follows from Proposition 2.10 that $d_{st}^2(G) = 1$. As $\delta(G) = 2$, Corollary 2.6 implies that $d_{st}(G) = 1$. □

Proposition 3.2. *If G is a wheel of order $n \geq 4$, then $d_{st}^3(G) = d_{st}^2(G) = d_{st}(G) = 1$.*

Proof. Since $N(v)$ contains a vertex of degree at most 3 for every vertex $v \in V(G)$, it follows from Proposition 2.10 that $d_{st}^3(G) = d_{st}^2(G) = 1$.

Now let $v_1, v_2, \ldots, v_n$ be the vertex set of the wheel G such that $v_1 v_2 \ldots v_{n-1} v_1$ is a cycle of length $n-1$ and v_n is adjacent to v_i for each $i = 1, 2, \ldots, n-1$. It follows from Proposition 2.3 that $d_{st}(G) \leq 3$, and thus Corollary 2.9 implies that $d_{st}(G) = 3$ or $d_{st}(G) = 1$. Suppose that $d_{st}(G) = 3$, and let $\{f_1, f_2, f_3\}$ be a signed total $(1,1)$-dominating family on G. If, without loss of generality, $f_1(v_n) = -1$, then we deduce that $f_1(v_1) = f_1(v_2) = \ldots = f_1(v_{n-1}) = 1$. Since $\sum_{x \in N(v_n)} f_i(x) \geq 1$ for $i = 2, 3$, it follows that $f_2(x) = 1$ and $f_3(x) = 1$ for at least $\lceil n/2 \rceil$ vertices $x \in \{v_1, v_2, \ldots, v_{n-1}\}$. Therefore $f_2(x) + f_3(x) = 2$ and thus $f_1(x) + f_2(x) + f_3(x) = 3$ for at least one vertex $x \in \{v_1, v_2, \ldots, v_{n-1}\}$. This is a contradiction to $f_1(x) + f_2(x) + f_3(x) \leq 1$ for each vertex $x \in V(G)$. Hence $d_{st}(G) = 3$ is impossible, and the proof is complete. □

The *cartesian product* $G = G_1 \times G_2$ of two vertex disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The cartesian product of two paths $P_r = x_1 x_2 \ldots x_r$ and $P_t = y_1 y_2 \ldots y_t$ is called a *grid*.

Proposition 3.3. *Let $G = P_r \times P_t$ be a grid of order $n = r \cdot t \geq 2$ such that $r \leq t$. Then $d_{st}(G) = 1$, $d_{st}^2(G) = 1$ when $r \leq 4$ and $d_{st}^2(G) = 2$ when $r \geq 5$.*

Proof. Since $\delta(G) = 2$, it follows from Proposition 2.3 and Corollary 2.9 that $d_{st}(G) = 1$.

If $r \leq 4$, then $N(v)$ contains a vertex of degree at most 3 for every vertex $v \in V(G)$, and so Proposition 2.10 implies that $d_{st}^2(G) = 1$.

If $r \geq 5$, then the vertex $v = (x_3, y_3) \in V(G)$ has the property that all vertices of $N(v)$ are of degree 4, and therefore we deduce from Proposition 2.10 that $d_{st}^2(G) \geq 2$. In view of Proposition 2.3, we see that $d_{st}^2(G) \leq \delta(G) = 2$ and thus $d_{st}^2(G) = 2$. □

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