

## ON THE EXISTENCE OF $k$ -KERNELS IN DIGRAPHS AND IN WEIGHTED DIGRAPHS

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### Abstract

Let  $D$  be a digraph,  $V(D)$  and  $A(D)$  will denote the sets of vertices and arcs of  $D$ , respectively.

A  $(k, l)$ -kernel  $N$  of  $D$  is a  $k$ -independent set of vertices (if  $u, v \in N$  then  $d(u, v), d(v, u) \geq k$ ) and  $l$ -absorbent (if  $u \in V(D) \setminus N$  then exists  $v \in N$  such that  $d(u, v) \leq l$ ). A  $k$ -kernel is a  $(k, k - 1)$ -kernel. We propose an extension of the definition of  $(k, l)$ -kernel to (arc-)weighted digraphs, verifying which of the existing results for  $k$ -kernels are valid in this extension. If  $D$  is a digraph and  $w : A(D) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $D$ , we can restate the problem of finding a  $k$ -kernel in the following way. If  $\mathcal{C}$  is a walk in  $D$ , the weight of  $\mathcal{C}$  is defined as  $w(\mathcal{C}) := \sum_{f \in A(\mathcal{C})} w(f)$ . A subset  $S \subseteq V(D)$  is  $(k, w)$ -independent if, for every  $u, v \in S$  there does not exist an  $uv$ -directed path of weight less than  $k$ . A subset  $S \subseteq V(D)$  will be  $(l, w)$ -absorbent if, for every  $u \in V(D) \setminus S$ , there exists an  $uS$ -directed path of weight less than or equal to  $l$ . A subset  $N \subseteq V(D)$  is a  $(k, l, w)$ -kernel if it is  $(k, w)$ -independent and  $(l, w)$ -absorbent. We prove, among other results, that every transitive digraph has a  $(k, k - 1, w)$ -kernel for every  $k$ , that if  $T$  is a tournament and  $w(a) \leq \frac{k-1}{2}$  for every  $a \in A(T)$ , then  $T$  has a  $(k, w)$ -kernel and that if every directed cycle in a quasi-transitive digraph  $D$  has weight  $\leq \frac{k-1}{2} + 1$ , then  $D$  has a  $(k, w)$ -kernel.

Also, we let the weight function to have an arbitrary group as codomain and propose another variation of the concept of  $k$ -kernel.

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### 1. Introduction

For general concepts and notation we refer the reader to [1, 3] and [6], particularly we will use the notation of [6] for walks, if  $\mathcal{C} = (x_0, x_1, \dots, x_n)$  is a walk and  $i < j$  then  $x_i \mathcal{C} x_j$  will denote the subwalk of  $\mathcal{C}$   $(x_i, x_{i+1}, \dots, x_{j-1}, x_j)$ , if  $x_i = x_0$  we will simply write  $\mathcal{C} x_j$ , analogously if  $x_j = x_n$ . Union of walks will be denoted by concatenation or with  $\cup$ .

In [11], M. Kwasnik introduces the concept of  $(k, l)$ -kernel in a digraph generalizing the concept of kernel of a digraph in the Berge's sense which is a  $(2, 1)$ -kernel. As a special

case of  $(k, l)$ -kernels we consider the  $k$ -kernels; we define a  $k$ -kernel to be a  $(k, k-1)$ -kernel. Under this definition a kernel is a 2-kernel.

There are not many results concerning the existence of  $k$ -kernels nor  $(k, l)$ -kernels in large families of digraphs, many of the existing results come from the study of products of digraphs and how the  $k$ -kernels are preserved (like the work of Włoch and Włoch, in particular with Szumny in [16, 17]) or the superdigraphs or certain families of digraphs ([10]); also sufficient conditions for a digraph to have a  $k$ -kernel can be found in [7].

Although the problem of finding sufficient conditions for a digraph to have  $k$ -kernel is a difficult one, and there is still a lot of work to do, we propose a variant that seems to be very interesting for the additional information that a digraph may hold when weights are assigned to its arcs as in many classical problems (*e.g.* minimum weight paths or spanning trees, flows in networks). This problem is even harder than the original one since it is an effective generalization, but we are confident that stating the problem and obtaining the first results will lead to new proof techniques and eventually to feedback to the original kernel and  $k$ -kernel problems.

We begin with some of the classical results in Kernel Theory that we will use as platform for the results we propose.

Since every (directed) cycle of odd length does not have a kernel, sufficient conditions for the existence of kernels in digraphs have been found by imposing conditions on the cycles of a digraph. For example in [15] it is proved that

**Theorem 1.1.** *If  $D$  is a digraph without directed cycles, then  $D$  has a kernel.*

In [13], Richardson generalizes this result as follows<sup>1</sup>.

**Theorem 1.2.** [13] *If  $D$  is a digraph such that the length of every directed cycle is congruent to 0 (mod 2) then  $D$  has a kernel.*

These two theorems are examples of results than can be generalized for  $k$ -kernels, our attention is focused in the generalization of the second theorem.

In [12], M. Kwaśnik stated the following generalization for  $k$ -kernels.

**Definition 1.3.** *A digraph  $D$  is strongly connected if and only if for every pair of vertices  $u, v \in V(D)$ , there exists a  $uv$ -directed path in  $D$ .*

*A digraph  $D$  is unilaterally connected, or simply unilateral, if and only if for every pair of vertices  $u, v \in V(D)$ , there exists an  $uv$ -directed path or a  $vu$ -directed path in  $D$ .*

**Theorem 1.4.** *Let  $D$  be a strongly connected digraph. If every directed cycle in  $D$  has length congruent to 0 (mod  $k$ ) then  $D$  has a  $k$ -kernel.*

In [14], an example is given to show that the strong connectedness in Theorem 1.4 is a necessary condition. In [8], the authors explore variants of Theorem 1.4, relaxing the

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<sup>1</sup>See [4] for a simpler proof of Theorem 1.2.

strong connectedness to unilaterality but restricting the length of certain cycles besides the directed ones to have a specific congruence modulo  $k$ . The idea came from the observation that the congruence to  $0 \pmod{k}$  together with the strong connectedness is a very strong combination, yielding a cyclically  $k$ -partite structure. The work developed in this paper arises in our attempt to obtain further resemblings Theorem 1.4. Our first try, reflected in Section 2, did not work so well in the direction of Theorem 1.4, but we won some insight of the weighted case mentioned in the abstract. In Section 3 we emphasized the congruence to 0 condition and realized that given a normal subgroup  $H$  of a group  $G$  it makes sense to think in the congruence modulo  $H$ , so an analog of Theorem 1.4 can be positively obtained.

## 2. Weighted digraphs

Kernels have been used to model mathematical objects, like minimal sets of counterexamples in finite theories, winning strategies in some games played on graphs or optimal sets of decisions. We think that  $(k, l)$ -kernels have potential to be used in real life models, *e.g.*, if a digraph represents the map of a city, a  $(k, l)$ -kernel is an optimum distribution of a service or good someone may offer to the population, according to the parameters  $k$  and  $l$  this distribution could be done choosing an appropriate distance between the service centers ( $k$ -independence) to avoid saturation in one zone and also an appropriate distance so all the population in the city have an easy access to the service ( $l$ -absorbence). In addition to this point of view, we want to add further information, not only the distance matters, but transportation involves additional costs, it may be time, or some toll, this information may be added by means of a weight function for the arcs of the digraph, so every arc would represent a distance unit and its weight the cost to cross it. So, our next aim is to generalize the concept of  $k$ -kernel adding weights to the arcs of the digraph and study the possible generalization for well known results on  $k$ -kernels.

Let us observe that if we want to find a  $(k, w)$ -kernel, an arc with weight greater than or equal to  $k$  will not contribute in any walk between vertices for independence nor absorbence; to avoid the case when there is an arc between two vertices and they remain  $(k, w)$ -independent or when a vertex cannot  $(k - 1, w)$ -absorb some of its in-neighbours, we will consider only weights between 1 and  $k - 1$  for the arcs. When  $w$  is the constant function equal to 1, a  $(k, l, w)$ -kernel is a  $(k, l)$ -kernel in Kwaśnik's sense, and as usual, a  $(k, k - 1, w)$ -kernel will be simply called  $(k, w)$ -kernel. Despite the fact that this definition generalizes efectively the notion of  $(k, l)$ -kernel, and thus the notion of  $k$ -kernel, many results do not remain true when  $w$  is not identically the consant 1.

For convenience we will say that the weighted distance from vertex  $u$  to vertex  $v$  with respect to the weight function  $w$  is the minimum weight of all the  $uv$ -directed paths, no matter the length. We will denote this as  $d^w(u, v)$ , as  $d(u, v)$  will denote the usual distance.

**Proposition 2.1.** *Theorem 1.4 is false for  $(k, w)$ -kernels.*

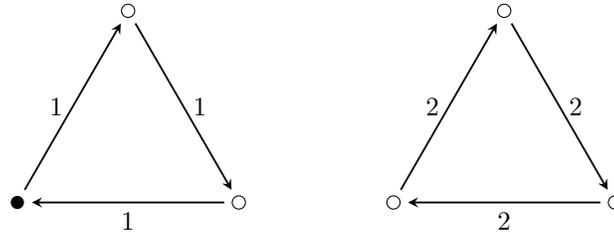


Figure 1: Figure of the left is  $C_3$  which have a 3-kernel (filled vertex) in the usual sense. On the right, a counterexample to the weighted version of Theorem 1.4.

*Proof.* In this digraph there is only one cycle of length  $6 \equiv 0 \pmod{3}$ , nonetheless has no 3-kernel. For every two vertices in this digraph one of the two weighted distances between them is two, then every maximal 3-independent set consists of exactly one vertex. But for every two vertices one of the two weighted distances between them is four, so for each maximal 3-independent set there exists a vertex that cannot be 2-absorbed.  $\square$

The next theorem, a generalization of a result due to Berge [3], about symmetrical digraphs is another example of a result that does not remain valid if a weight function is considered.

**Theorem 2.2.** *If  $D$  is a symmetrical digraph and  $k \geq 2$  then every maximal  $k$ -independent subset of vertices is a  $k$ -kernel.*

**Proposition 2.3.** *Theorem 2.2 is false for  $(k, w)$ -kernels.*

*Proof.* The digraph on Figure 2 is a counterexample.

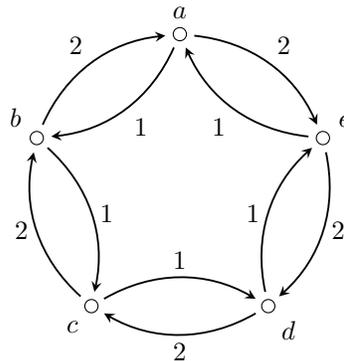


Figure 2: Counterexample to the weighted version of Theorem 2.2.

The set  $\{a\}$  is a maximal 3-independent subset of vertices since vertices  $b$  and  $c$  are at weighted distance one and two, respectively, from  $a$  and vertices  $e$  and  $d$  are at weighted

distance one and two, respectively, to  $a$ . However, vertex  $a$  is at weighted distance three from vertex  $c$ , thus  $c$  cannot be  $(2, w)$ -absorbed by  $a$ .  $\square$

Note that the digraph in Figure 2 does not have a 3-kernel, so we go beyond the statement of Theorem 2.2, it is not only that maximal  $k$ -independent sets in symmetrical digraphs are not  $k$ -kernels, there are symmetrical digraphs without a  $k$ -kernel. However, there is a simple case when a symmetrical digraph has a kernel for every  $k \in \mathbb{N}$ . Also, and just as a curious observation, the weight assignment for the arcs of the digraph in Figure 2 conform a nowhere-zero 3-flow for the given digraph.

**Theorem 2.4.** *If  $D$  is a symmetrical digraph and  $w : A(D) \rightarrow \mathbb{Z}$  is a constant weight function for the arcs of  $D$ , then every maximal  $(k, w)$ -independent subset of  $V(D)$  is a  $(k, w)$ -kernel for every  $k \geq 2$ .*

*Proof.* First, let us observe that for  $k = 2$ , since every arc have weight  $\leq k - 1$ , the result is the classical result for kernels in symmetrical non-weighted digraphs. The proof is analogous to the proof of the original result. Let  $N$  be a maximal  $(k, w)$ -independent subset of  $V(D)$ . If  $N$  is  $(k - 1, w)$ -absorbent, then  $N$  is the desired  $(k, w)$ -kernel. So, let us assume that there exists a vertex  $v \in V(D)$  such that it is not  $(k - 1, w)$ -absorbed by  $N$ . Then  $d^w(v, N) \geq k$ , but  $D$  is symmetrical and  $w$  is a constant function, so  $d^w(N, v) \geq k$  and  $N \cup \{v\}$  is  $(k, w)$ -independent, contradicting the choice of  $N$  as a maximal  $(k, w)$ -independent set.  $\square$

Figure 1 shows that result of Theorem 2.4 is not as obvious as it may seem, there are other theorems that become invalid even in the constant weights case. Also, the hypothesis of Theorem 2.4 is not tight, there are symmetrical digraphs with non constant weight functions and  $k$ -kernel as the example in Figure 3 shows; sets  $\{b\}$  and  $\{c\}$  are both 3-kernels for the digraph.

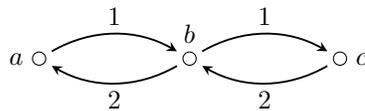


Figure 3: Symmetrical digraph with 3-kernel and non constant weight function.

**Theorem 2.5.** *If  $D$  is an acyclic digraph and  $w : A(D) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $D$ , then  $D$  has a unique  $(k, w)$ -kernel for each  $k \geq 2$ .*

*Proof.* Let us proceed by induction on  $|V(D)|$  with fixed  $k \geq 2$ . If  $|V(D)| = 1$ , the only vertex of  $D$  is the desired  $(k, w)$ -kernel. Assuming the result valid for every acyclic digraph  $D$  such that  $|V(D)| < n$ , let  $D$  be an acyclic digraph with  $|V(D)| = n$ . Since  $D$  is an acyclic digraph, there exists  $v \in V(D)$  such that  $d^-(v) = 0$ . Now,  $D - v$  is an acyclic digraph on

$n - 1$  vertices and by induction hypothesis has a unique  $(k, w)$ -kernel  $N'$ . There are two cases:

**Case 1.** If  $v$  is  $(k - 1, w)$ -absorbed by  $N'$  in  $D$ , then  $N'$  is the  $(k - 1, w)$ -kernel we have been looking for.

**Case 2.** If  $v$  is not  $(k - 1, w)$ -absorbed by  $N'$  in  $D$ , then there does not exist  $vN'$ -directed path of weight less than or equal to  $k - 1$  and, as  $v$  has indegree 0 there does not exist  $N'v$ -directed path in  $D$ , in particular there does not exist  $N'v$ -directed path of weight less or equal than  $k - 1$  and hence  $N = N' \cup \{v\}$  is  $(k, w)$ -independent and  $(k - 1, w)$ -absorbent in  $D$ . We have found in  $N$  the desired  $(k, w)$ -kernel.

Finally, observe that in either case  $N'$  is unique by inductive hypothesis. If  $M$  is a  $(k, w)$ -kernel for  $D$ ,  $M \setminus \{v\}$  is  $(k, w)$ -independent in  $D - v$  and, as  $d_D^-(v) = 0$ ,  $v$  cannot  $(k - 1, w)$  absorb any other vertex, therefore  $M \setminus \{v\}$  is  $(k - 1, w)$ -absorbent in  $D - v$  and a  $(k, w)$ -kernel of  $D - v$ . It follows that  $M \setminus \{v\} = N \setminus \{v\}$  and hence  $M = N$ , the unique  $(k, w)$ -kernel of  $D$ .  $\square$

If we extend the notion of diameter to match with our new weighted distance, we can derive some results, although rather simple, we have seen that other results become invalid in the weighted versions.

**Lemma 2.6.** *If  $D$  is a digraph,  $w : A(D) \rightarrow \mathbb{Z}$  a weight function for  $A(D)$  and under this function  $D$  has weighted diameter less than or equal to  $k - 1$ , then every vertex of  $D$  is a  $(h, w)$ -kernel for each  $k \leq h$ .*

*Proof.* Let  $v \in V(D)$ ,  $h \geq k$  and  $w : A(D) \rightarrow \mathbb{Z}$  a weight function for the arcs of  $D$ . Clearly, as  $\{v\}$  has only one vertex, is a  $(h, w)$ -independent set. Since  $D$  has diameter less than or equal to  $k - 1$  then, for every  $u \in V(D)$  if  $\mathcal{C}$  is a  $uv$ -directed path,  $\sum w(f)_{f \in A(\mathcal{C})} \leq k - 1 \leq h - 1$ , thus  $\{v\}$  is a  $(h - 1, w)$ -absorbent set and consequently a  $(h, w)$ -kernel.  $\square$

We will introduce a definition that has proved to be very useful in kernel theory, the weighted case is no exception and we will use it in many proofs in the rest of this work.

**Definition 2.7.** *If  $D$  is a digraph, the condensation digraph of  $D$  or simply the condensation of  $D$  is the digraph  $D^*$  which vertices are the strong components of  $D$ ,  $\{D_1, D_2, \dots, D_p\}$  and  $(D_i, D_j) \in A(D^*)$  if and only if there exist a  $D_i D_j$ -arc in  $D$ .*

It is direct to observe that  $D^*$  has no directed cycles, and so, for every digraph  $D$ , the condensation  $D^*$  has at least one vertex of indegree 0 and one vertex of exdegree 0, these will be called initial and terminal components of  $D$  (or vertices of  $D^*$ ) respectively.

**Theorem 2.8.** *If  $D$  is a digraph,  $w : A(D) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $D$  and every strong component of  $D$  has diameter less than or equal to  $k - 1$  under the weight function, then  $D$  has a  $(k, w)$ -kernel.*

*Proof.* By induction on the number of strong components of  $D$ . If  $D$  has a unique strong component, then  $D$  is strongly connected and has diameter less than or equal to  $k - 1$ . In virtue of Lemma 2.6 any vertex of  $D$  is a  $(k, w)$ -kernel for  $D$ .

Let  $D$  be a digraph with  $\varphi$  strong components of diameter less than or equal to  $k - 1$  and suppose for inductive hypothesis the theorem valid for every digraph with less than  $\varphi$  strong components. Since the condensation digraph  $D^*$  of  $D$  has no directed cycles we can consider an initial strong component of  $D$ , say  $C_0$ . The digraph  $D - C_0$  has  $\varphi - 1$  strong components and all of its components have diameter less than or equal to  $k - 1$ . As a consequence of the inductive hypothesis  $D - C_0$  has  $(k, w)$ -kernel  $N'$ . It is easy to observe that  $N'$  is  $(k, w)$ -independent not only in  $D - C_0$  but in all  $D$  because  $C_0$  is an initial component and there are no  $(D - C_0)C_0$ -paths in  $D$  and therefore there are no new  $N'N'$ -paths in  $D$ , so if  $N'$  is  $(k - 1, w)$ -absorbent in  $D$ , then  $N'$  is a  $(k, w)$ -kernel of  $D$ , if not, there exists a vertex  $v \in V(C_0)$  not  $(k - 1, w)$ -absorbed by  $N'$  and hence, there are not  $vN'$ -directed paths with weight less than or equal to  $k - 1$  in  $D$ . Being  $C_0$  an initial component there does not exist  $N'v$ -directed path in  $D$ , so  $N = N' \cup \{v\}$  is a  $(k, w)$ -independent set in  $D$ . Is easy to observe that  $N$  is  $(k - 1, w)$ -absorbent set for  $D$ , this is because  $N'$   $(k - 1, w)$ -absorbs all vertices in  $D - C_0$  and, as a result of Lemma 2.6,  $\{v\}$   $(k - 1, w)$ -absorbs all vertices in  $C_0$ . Consequently,  $N$  is a  $(k, w)$ -kernel for  $D$  and the desired result follows from the induction principle.  $\square$

The next Lemma and its consequences take advantage of the existing results for  $(k, l)$ -kernels, in some cases those results can be adapted with hypothesis which are not very restrictive to fit in the weighted case.

**Lemma 2.9.** *Let  $D$  be a digraph and  $w : A(D) \rightarrow \mathbb{Z}$  a weight function. If  $N$  is a  $(k, l)$ -kernel for  $D$ , and for every directed path  $\mathcal{C}$  of length less than or equal to  $l$ , the condition  $w(\mathcal{C}) \leq k - 1$  holds, then  $N$  is a  $(k, w)$ -kernel for  $D$ .*

*Proof.* Since  $N$  is  $k$ -independent, the weighted distance between vertices of  $N$  is greater than or equal to  $k$ , and thus  $N$  is  $(k, w)$ -independent. Also,  $N$  is  $l$ -absorbent, so for a vertex  $v \in V(D) \setminus N$  there exists a  $vN$ -directed path  $\mathcal{C}$  of length  $\leq l$ , but for hypothesis,  $w(\mathcal{C}) \leq k - 1$ , therefore  $v$  is  $(k - 1, w)$ -absorbed by  $N$ .  $\square$

We have two direct applications of Lemma 2.9 for cases  $l = 1$  and  $l = 2$ , we just have to recall a proposition and a theorem, the first concerning the structure of transitive digraphs, the second stating a general result about  $(k, l)$ -kernels in tournaments.

**Definition 2.10.** *Let  $D$  be a digraph with  $V(D) = \{v_1, v_2, \dots, v_n\}$  and let  $G_1, G_2, \dots, G_n$  be pairwise vertex-disjoint digraphs. The composition  $D[G_1, G_2, \dots, G_n]$  is the digraph  $L$  with  $V(L) = \bigcup_{i=1}^n (V(G_i))$  and  $A(L) = (\bigcup_{i=1}^n F(G_i)) \cup \{g_i g_j \mid g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$ .*

The next proposition is left as an exercise in [1].

**Proposition 2.11.** *Let  $D$  be a digraph with an acyclic ordering  $D_1, D_2, \dots, D_p$  of its strong components. The digraph  $D$  is transitive if and only if each of  $D_i$  is complete, the digraph  $D^*$  obtained from  $D$  by contraction of  $D_1, \dots, D_p$  followed by deletion of multiple arcs is a transitive oriented graph, and  $D = D^*[D_1, D_2, \dots, D_p]$ , where  $p = |V(D^*)|$ .*

Observe that the digraph  $D^*$  in Proposition 2.11 is the condensation digraph of  $D$ .

**Theorem 2.12.** *If  $D$  is a transitive digraph and  $w : A(D) \rightarrow \mathbb{Z}$  is a weight function, then  $D$  has a  $(k, w)$ -kernel.*

*Proof.* We just have to observe the structure of a  $k$ -kernel in a transitive digraph. Since every transitive digraph is the composition of an acyclic transitive digraph  $D^*$  (the condensation of  $D$ ) and complete digraphs  $D_1, D_2, \dots, D_p$ , a  $k$ -kernel in a transitive digraph can be constructed choosing one vertex in every terminal strong component of  $D$  (vertices with exdegree 0 in  $D^*$ ). In virtue of Proposition 2.11, such  $k$ -kernel will be not only  $k$ -independent but independent by directed paths, and not only  $(k - 1)$ -absorbent, but 1-absorbent. So if  $N$  is a  $k$ -kernel for a transitive digraph  $D$ , then  $N$  is a  $(k, 1)$ -kernel in  $D$  and the result follows from Lemma 2.9.  $\square$

The next theorem is a classical result for tournaments, can be found in [5].

**Theorem 2.13.** *Every tournament has a 2-absorbent vertex.*

As an immediate consequence of Theorem 2.13, every tournament has a  $(k, 2)$ -kernel for every  $k \geq 2$ , the set containing the 2-absorbent vertex will work.

**Theorem 2.14.** *If  $T$  is a tournament and  $w : A(T) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $T$  such that every directed path of length 2 has weight less than or equal to  $k - 1$ , then  $T$  has a vertex  $v$  such that  $\{v\}$  is a  $(k, w)$ -kernel.*

*Proof.* It follows directly from Lemma 2.9 and Theorem 2.13.  $\square$

**Corollary 2.15.** *If  $T$  is a tournament and  $w : A(T) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $T$  such that  $w(a) \leq \frac{k-1}{2}$  for all  $a \in A(T)$ , then  $T$  has a  $(k, w)$ -kernel.*

*Proof.* Let  $(x, y, z)$  be a directed path, then  $w(x, y, z) = w(x, y) + w(y, z) \leq \frac{k-1}{2} + \frac{k-1}{2} = k - 1$ . The result now follows from Theorem 2.14.  $\square$

Additionally we will need a couple of structural results for the non-quasitransitive strong digraphs. The next proposition and theorem, due to Bang-Jensen and Huang, can be found in [2].

**Proposition 2.16.** *Let  $D$  be a quasi-transitive digraph. Suppose that  $P = (x_1, x_2, \dots, x_k)$  is a minimal  $x_1 x_k$ -path. Then the subdigraph induced by  $V(P)$  is a semicomplete digraph and  $(x_j, x_i) \in A(D)$  for every  $2 \leq i + 1 < j \leq k$ , unless  $k = 4$ , in which case the arc between  $x_1$  and  $x_k$  may be absent.*

**Theorem 2.17.** *Let  $D$  be a digraph which is quasi-transitive. If  $D$  is not strong, then there exist a transitive oriented graph  $T$  with vertices  $u_1, u_2, \dots, u_t$ , and strong quasi-transitive digraphs  $H_1, H_2, \dots, H_t$  such that  $D = T[H_1, H_2, \dots, H_t]$ .*

Once again, it is clear that  $T$  is the condensation of  $D$ .

In [9], the authors have proved the next theorem.

**Theorem 2.18.** *Every quasi-transitive digraph have a  $k$ -kernel for every  $k \geq 3$ .*

In the proof of this theorem the technique is again to choose a  $k$ -kernel for each of the terminal strong components of the given digraph.

**Theorem 2.19.** *If  $k \geq 3$ ,  $D$  is a quasi-transitive digraph and  $w : A(D) \rightarrow \mathbb{Z}$  is a weight function for the arcs of  $D$  such that every directed cycle has weight less than or equal to  $\lfloor \frac{k-1}{2} \rfloor + 1$ , then  $D$  has a  $k$ -kernel.*

*Proof.* We know that every quasi-transitive digraph has a  $k$ -kernel for every  $k \geq 3$ . Let  $N$  be a  $k$ -kernel for  $D$ . We will prove that  $N$  is a  $(k, w)$ -kernel. It is clear that  $N$  is  $(k, w)$ -independent in  $D$  because it is  $k$ -independent and the weighted distance is greater than or equal to the distance. For the  $(k - 1, w)$ -absorbence let  $v \in V(D) \setminus N$  be a vertex not in  $N$ . If  $v$  is not in a terminal strong component of  $D$ , then, by Theorem 2.17 and the observation to Theorem 2.18,  $v$  is absorbed at distance 1 by  $N$ , so it is  $(k - 1, w)$ -absorbed by  $N$ . If  $v$  is in a terminal strong component  $S$  of  $D$ , then there is at least one vertex  $u \in V(S)$  such that  $u \in N$  and  $u$   $(k - 1)$ -absorbs  $v$ , so we have three possibilities, that  $d(u, v) = 1$ ,  $d(u, v) \notin \{1, 3\}$  and  $d(u, v) = 3$ .

**Case 1.** If the non-weighted distance from  $u$  to  $v$  is 1, then, as  $S$  is strong, there must be a  $vu$ -directed path  $\mathcal{C}$  and together with the arc  $(u, v)$  it forms a directed cycle, wich by hypothesis must have total weight  $\leq \lfloor \frac{k-1}{2} \rfloor + 1$ , so  $w(\mathcal{C}) \leq \frac{k-1}{2}$ , and hence  $v$  is  $(k - 1, w)$ -absorbed by  $N$ .

**Case 2.** If the non-weighted distance from  $u$  to  $v$  is not 1 nor 3, then in virtue of Proposition 2.16,  $v$  is 1-absorbed by  $N$ , and then, since the arc weights are bounded by  $k - 1$ ,  $v$  is  $(k - 1, w)$ -absorbed by  $N$ .

**Case 3.** If the non-weighted distance from  $u$  to  $v$  is 3, then there exists a  $uv$ -directed path  $\mathcal{C} = (u, x, y, v)$  and we have several subcases. Observe that  $d(v, u) \in \{1, 3\}$ , otherwise, as  $D$  is quasi-transitive, Proposition 2.16 would imply that  $d(u, v) = 1$  which contradicts the assumption for this case.

**Case 3.1.** If  $(v, u) \in A(D)$ , then  $v$  is  $(k - 1, w)$ -absorbed by  $N$  because  $w(v, u) \leq k - 1$ .

**Case 3.2.** If  $d(v, u) = 3$  then one of the subcases depicted in Figure 5 occurs.

**Case 3.2.a** Because of the weight restriction for the directed cycles, the two cycles  $C_1 = (u, x, y, u)$  and  $C_2 = (x, y, v, x)$  fulfill  $w(C_1), w(C_2) \leq \lfloor \frac{k-1}{2} \rfloor + 1$ , but  $w(C_1) = p_1 + p_2 + q_2$  and  $w(C_2) = p_2 + p_3 + q_1$ , adding this inequalities we have

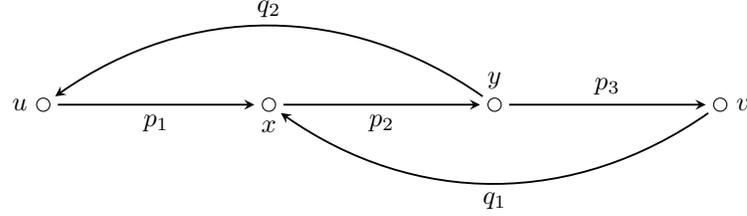


Figure 4: Case 3.2.a.

$$\begin{aligned}
 p_1 + 2p_2 + p_3 + q_1 + q_2 &\leq 2 \left\lfloor \frac{k-1}{2} \right\rfloor + 2 \\
 q_1 + p_2 + q_2 &\leq 2 \left\lfloor \frac{k-1}{2} \right\rfloor + 2 - (p_1 + p_2 + p_3) \\
 &\leq 2 \left\lfloor \frac{k-1}{2} \right\rfloor + 2 - 3 \\
 &= 2 \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \\
 &< k - 1.
 \end{aligned}$$

So,  $w(v, x, y, u) = q_1 + p_2 + q_2 < k - 1$  and thus,  $v$  is  $(k - 1, w)$ -absorbed by  $N$ .

**Case 3.2.b** Since  $d(u, v) = 3$ , the quasi-transitivity and the existence of the arcs  $(u, x), (x, y), (y, v)$  implies that  $(y, u), (v, x) \in A(D)$ , which reduces this case to case 3.2.a.

**Case 3.2.c** This case cannot occur, since  $(u, x), (x, y) \in A(D)$ , by the quasi-transitive hypothesis, then  $(u, y)$  or  $(y, u)$  must be in  $A(D)$ , but  $d(u, v) = d(v, u) = 3$ , and  $(u, y)$  together with  $(y, v)$  would imply that  $d(u, v) = 2$ , also  $(y, u)$  together with  $(v, y)$  would imply that  $d(v, u) = 2$ , as a contradiction arise in both cases, this case is impossible.

**Case 3.2.d** The same argument used in Case 3.2.c shows that this case cannot happen.

**Case 3.2.e** Since  $(x, y), (y, v) \in A(D)$ , by the quasi-transitive hypothesis,  $(x, v)$  or  $(v, x)$  must be in  $A(D)$ , but analogous to Case 3.2.c,  $(u, x)$  together with  $(x, v)$  would imply that  $d(u, v) = 2$ , also  $(v, x)$  together with  $(x, u)$  would imply that  $d(v, u) = 2$ , as a contradiction arise in both cases, this case is impossible.

**Case 3.2.f** This configuration can be reduced to the one in Case 3.2.a, observe that  $(u, x), (x, y) \in A(D)$  and, as  $d(u, v) = 3$ ,  $(y, u) \in A(D)$ , therefore we have the same configuration as in the mentioned case.

**Case 3.2.g** This case can also be reduced to Case 3.2.a, we just need the existence of the arc  $(v, x)$ , which is justified by the existence of the arcs  $(x, y), (y, v)$ , the quasi-transitive hypothesis and the fact that  $d(u, v) = 3$ , and hence  $(x, v) \notin A(D)$ .

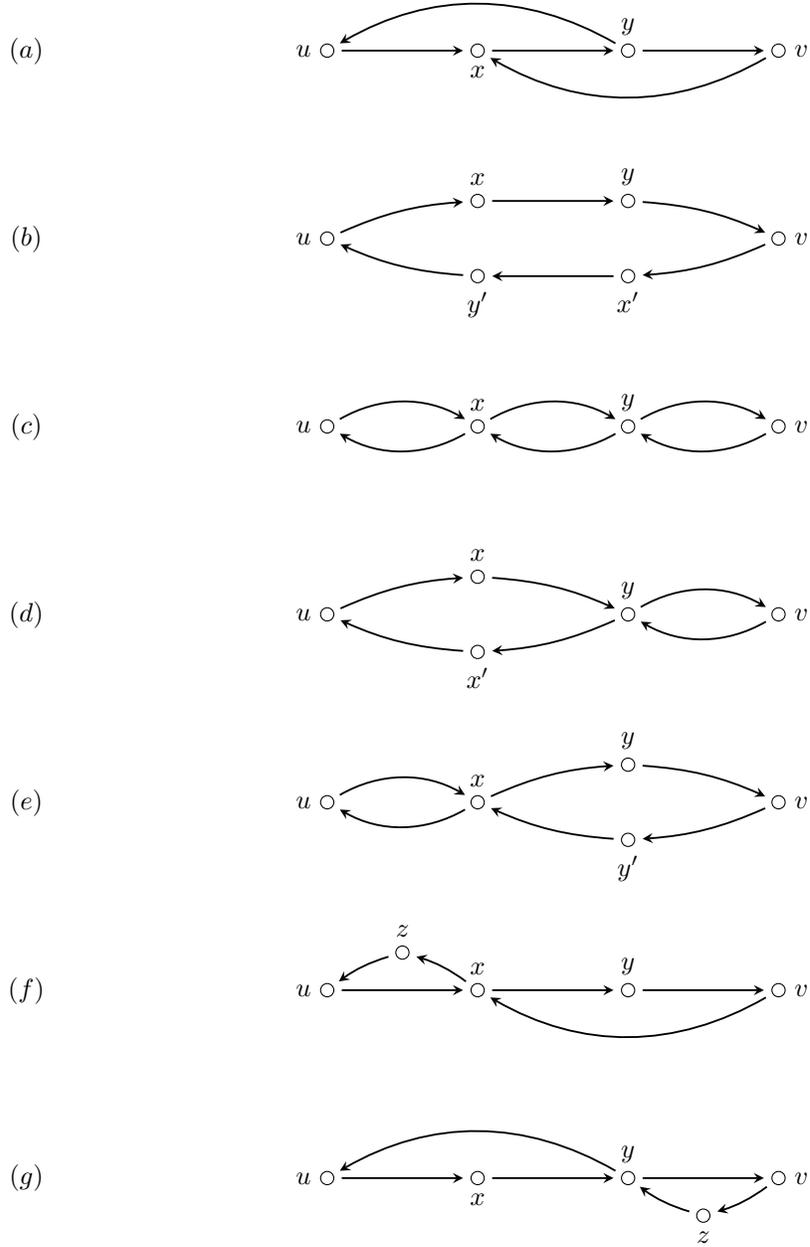


Figure 5: Subcases for Case 3.4 in the proof of Theorem 2.19.

Since the cases are exhaustive, we can conclude that every vertex in  $V(D)$  is  $(k - 1, w)$ -absorbed by  $N$ . Thus, as  $N$  is  $(k, w)$ -independent, it is the  $(k, w)$ -kernel we have been looking for.  $\square$

The hypothesis of Theorem 2.19 are sufficient but not necessary. Figure 6 shows a digraph with directed cycles of weight greater than  $\lfloor \frac{5-1}{3} \rfloor + 1 = 2$  and a 5-kernel. Cycle  $(a, d, b, c, a)$  has weight 8 and cycle  $(a, b, c, a)$  has weight 6; however, sets  $\{b\}$ ,  $\{c\}$  and  $\{d\}$  are 5-kernels for the digraph. Also, we do not know if the bound for the weight of the cycles is tight, we were unable to find an example in which the equality is necessary, fact that make us think that the bound could be improved.

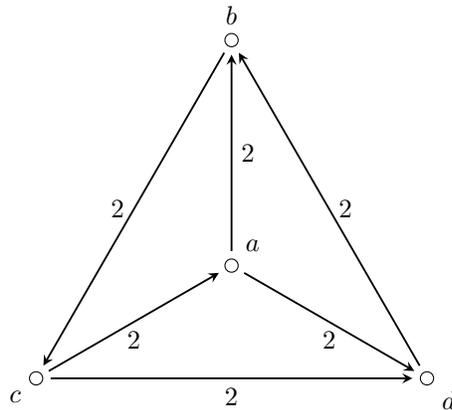


Figure 6: A digraph with directed cycles of weight greater than  $\lfloor \frac{5-1}{3} \rfloor + 1 = 2$  and a 5-kernel.

Finally, let us observe that Figure 1 is also a counterexample for the assertion that every quasi-transitive digraph has a  $(k, w)$ -kernel.

### 3. Digraphs with group weights

As many results that are valid in the non-weighted case does not remain valid in the weighted case, we have that the problem of finding a  $(k, w)$ -kernel in a given digraph is vastly more complicated than the non-weighted one, so inspired by Theorem 1.4 and in view of the great difficulty that the integer (or even natural)-valued functions represent, we considered the weight function to have an arbitrary group for codomain since a group is the simplest algebraic structure where a definition of congruence exists and we think that the key to Theorem 1.4 is the congruence modulo 0 condition. Let us recall that if  $G$  is a group and  $H$  is a subgroup of  $G$ , then if  $g, h \in G$ ,  $g \equiv h \pmod{H}$  if and only if  $gh^{-1} \in H$ . Nonetheless, as an arbitrary group has no order, we cannot state a direct analogy between a walk's weight and its length, thus, former results cannot be formally generalized, but an interesting result can be stated ressem bling Theorem 1.4.

**Definition 3.1.** Let  $D$  be a digraph,  $G$  a group and  $w : A(D) \rightarrow G$  a weight function for the arcs of  $D$ .

If  $H$  is a subgroup of  $G$  we will say that a walk  $\mathcal{P} = (v_0, v_1, \dots, v_n)$  has weight in  $H$ , if  $\sum_{i=0}^{n-1} w(v_i v_{i+1}) = w(\mathcal{P}) \in H$ .

**Definition 3.2.** Let  $H$  be a subgroup of  $G$ . A subset  $S \subseteq V(D)$ , is  $H$ -independent if for all  $u, v \in S$  there does not exist an  $uv$ -path with weight in  $H$  nor a  $vu$ -path with weight in  $H$ . We will say that  $S$  is  $H$ -absorbent if for each  $u \in V(D) \setminus S$  there exists a  $uS$ -path of weight in  $H$ . A  $H$ -independent,  $H$ -absorbent set will be a  $H$ -kernel.

**Example 3.3.** If  $D$  is a digraph, and we let  $G = \mathbb{Z}$ ,  $H = n\mathbb{Z}$  and  $w \equiv 1$ , then a  $(n\mathbb{Z})$ -kernel will be a subset  $N \subseteq V(D)$  such that between every pair of vertices of  $N$  there are no walks of length a multiple of  $n$  and from every vertex in  $V(D) \setminus N$  there is a walk in  $D$  of length a multiple of  $n$ .

**Lemma 3.4.** If  $D$  is a digraph,  $G$  a group,  $H$  a normal subgroup of  $G$  and  $w : A(D) \rightarrow G$  a weight function for  $A(D)$  such that every directed cycle  $\mathcal{C}$  fulfills  $w(\mathcal{C}) \in H$ , then every  $uv$ -directed walk with weight in  $H$  contains a  $uv$ -directed path with weight in  $H$ .

*Proof.* Before beginning the proof we want to point out that,  $H$  being a normal subgroup of  $G$  it does not matter in which vertex we begin summing the weight of a cycle (or, in other words, where we begin crossing the cycle), for if the arcs of the cycle are  $a_1, a_2, \dots, a_k$  with weights  $b_1, b_2, \dots, b_k$  respectively then, for the normality of  $H$ , if  $b_1 + b_2 + \dots + b_k \in H$  also  $-b_1 + b_1 + b_2 + \dots + b_k + b_1 \in H$ , hence for every cyclic permutation of the indices the sum of the weights of the cycle is in  $H$ . We proceed with the proof by induction on the length of the  $uv$ -directed walk. If the length is 1, then the walk is a path and the base case of the induction follows. Let's assume the result valid for every  $uv$ -directed walk of length strictly less than  $n$  and let  $\mathcal{R} = (u = x_0, x_1, \dots, x_{n-1}, x_n = v)$  be a  $uv$ -directed walk of length  $n$ . If  $x_i \neq x_j$  for each  $i \neq j$  then  $\mathcal{R}$  is a path. If not, then exists a cycle  $\mathcal{C} = (x_i, x_{i+1}, \dots, x_{k-1}, x_k = x_i)$  contained as a subsequence of  $\mathcal{R}$ . Let  $\mathcal{R}' = (x_0, \dots, x_i)$  and  $\mathcal{R}''(x_{k+1}, \dots, x_n)$ , then  $\mathcal{R} = \mathcal{R}'\mathcal{C}\mathcal{R}''$  therefore  $w(\mathcal{R}) = w(\mathcal{R}') + w(\mathcal{C}) + w(\mathcal{R}'')$ . For the theorem hypothesis  $w(\mathcal{C}) \in H$  and  $w(\mathcal{R}) \in H$ , since  $H$  is normal in  $G$ , it follows that  $-w(\mathcal{R}') + w(\mathcal{R}) + w(\mathcal{R}') = w(\mathcal{C}) + w(\mathcal{R}'') + w(\mathcal{R}) \in H$  and after a few simple calculations  $w(\mathcal{R}'') + w(\mathcal{R}') \in H$ ; using the normality of  $H$  once again we obtain  $-w(\mathcal{R}'') + w(\mathcal{R}'') + w(\mathcal{R}') + w(\mathcal{R}'') = w(\mathcal{R}') + w(\mathcal{R}'') \in H$ . Therefore  $\mathcal{R}'\mathcal{R}''$  is a  $uv$ -directed walk with weight in  $H$  of length strictly less than  $n$ , the desired result follows from the induction hypothesis.  $\square$

The next theorem is inspired by Theorem 1.4.

**Theorem 3.5.** If  $D$  is a digraph,  $G$  a group,  $H$  a normal subgroup of  $G$  and  $w : A(D) \rightarrow G$  is a weight function for the arcs of  $D$  such that every directed cycle of  $D$  has weight in  $H$ , then  $D$  has  $H$ -kernel.

*Proof.* Let  $N_0$  be a maximal  $H$ -independent subset of  $V(D)$  such that if  $\text{Abs}(N_0)$  is the set of vertices  $H$ -absorbed by  $N_0$ , then  $|\text{Abs}(N_0) \cup N_0|$  is maximum. If  $V(D) - (N_0 \cup \text{Abs}(N_0)) = \emptyset$ , then  $N_0$  is a  $H$ -kernel for  $D$ . If  $V(D) - (N_0 \cup \text{Abs}(N_0)) \neq \emptyset$ , then exists

a vertex  $v_0$  that is not in  $N_0$  nor  $H$ -absorbed by it. Since  $N_0$  is maximal  $H$ -independent and  $v_0$  is not  $H$ -absorbed by  $N_0$ , there exists in  $D$  a  $N_0v_0$ -directed path with weight in  $H$ . Let  $A \subseteq N_0$  be the set of vertices in  $N_0$  that are the initial vertex of a directed path with final vertex  $v_0$  and weight in  $H$  and let  $N_1 = (N_0 - A) \cup \{v_0\}$ . Clearly  $N_1$  is  $H$ -independent because  $N_0$  was, and as we added  $v_0$  we removed from  $N_0$  all the vertices that reached  $v_0$  with a directed path with weight in  $H$ , besides there does not exist any  $v_0N_0$ -directed path with weight in  $H$  in  $D$ . Also,  $\text{Abs}N_0 \subseteq \text{Abs}N_1$ ;  $N_1H$ -absorbs every vertex in  $A$  and, if  $x$  is a vertex in  $\text{Abs}(N_0)$  absorbed by  $A$ , then an  $xa$ -directed path of weight in  $H$  exists for some  $a \in A$ , but we have the existence of an  $av_0$ -directed path with weight in  $H$  so, there exists a  $xv_0$ -directed walk with weight in  $H$  and, by Lemma 3.4, there is a  $xv_0$ -directed path with weight  $H$  in  $D$ , which finally results in the  $H$ -absorption of  $x$  by  $N_1$ . Now,  $|\text{Abs}(N_0) \cup N_0| < |\text{Abs}(N_1) \cup N_1|$ , since  $N_1$   $H$ -absorbs at least one more vertex than  $N_0$ , that is to say  $v_0$ , and  $\text{Abs}(N_0) \subseteq \text{Abs}(N_1)$ . Nonetheless this results in a contradiction to the choice of  $N_0$  as a set that maximizes  $|\text{Abs}(N_0) \cup N_0|$ . So,  $N_0$  is the  $H$ -kernel we have been looking for.  $\square$

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