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On a problem of Domke, Dunbar, Haynes, Hedetniemi, and Markus Concerning the inverse domination number

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Abstract

The problem of the title is: can the inverse domination number of a finite simple graph with no isolated vertices ever exceed its vertex independence number? We consider a related problem: which graphs contain a minimum dominating set and a maximal independent set which are disjoint? Main results: among graphs with domination number ≤ 4, the answer to the first question is no, and among graphs with domination number ≤ 2, there is such a pair of sets as called for in the second question unless the graph is $K_{m,n}$ for some $m, n \geq 3$.

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1. Introduction

Throughout, G will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $v \in V$, $N_G(v) = \{u \in V : uv \in E\}$ and $N_G[v] = \{v\} \cup N_G(v)$. If G is the only graph in the discussion, N will replace N_G. If $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and

$N[S] = S \cup N(S)$. A set $S \subseteq V$ is dominating (in G) if $V = N[S]$. The *domination number* of G is $\gamma(G)$= min[$|S| : S \subseteq V$ is dominating].

A minimum dominating set in G is a dominating set $S \subseteq V$ such that $|S| = \gamma(G)$. A minimal dominating set is a dominating set no proper subset of which is a dominating set. It is easy to see that if G has no isolated vertices and S is a minimal dominating set in G, then $V \setminus S$ is dominating in G. When G has no isolated vertices, the *inverse domination number* of G is $\gamma'(G)$=min[$|B|$: $B \subseteq V \setminus S$ for some minimum dominating set $S \subseteq V$, and B is dominating in G.]

A set $I \subseteq V$ is *independent* if no two vertices of I are adjacent in G. A maximal independent set is one not properly contained in any other independent set. Clearly a maximal independent set in G is dominating in G. The (upper) *independence number* $\alpha(G)$ and the *lower independence number* $\imath(G)$ are defined by $\alpha(G)$ = max[$|I|$: $I \subseteq V$ is independent] and $\imath(G)$ = min[$|I|$: $I \subseteq V$ is maximal independent]. By remarks above, clearly $\gamma(G) \leq \imath(G) \leq \alpha(G)$.

The inverse domination number was introduced by Kulli and Sigarkanti [6], who showed that $\gamma'(G)$ is well defined when G has no isolated vertices. They went on to assert that when G has no isolated vertices then $\gamma'(G) \leq \alpha(G)$. Their attempted proof of this assertion was invalid; this was noticed, in due course, by a number of people including Gayla Domke, Jean Dunbar, Teresa Haynes, Steve Hedetniemi, and Lisa Markus, who all had a hand in transmitting the open question to the world. A second paper on the inverse domination number was published by Domke et al. [1]. The first and third authors heard about the problem from Haynes, Hedetniemi, and Markus at a conference at Clemson University in 2000 or 2001. Hedetniemi offered a prize for a resolution of the problem: a copy of [2].

Observe that if $G = K_{1,t}$ then $\imath(G) = 1 = \gamma(G)$ and $\alpha(G) = t = \gamma'(G)$, a sign that this problem may be more difficult than might have been evident at first glance.

We will say that G has Property DI if there exists a minimum dominating set $D \subseteq V$ and a maximal independent (and therefore dominating) set $I \subseteq V \setminus D$. If such a D and I exist, then straight from the definitions we have $\gamma'(G) \leq |I| \leq \alpha(G)$. [It is easy to see that if G has property DI then G has no isolated vertices – any isolated vertex would have to be in every dominating set.] Therefore, a graph G such that $\alpha(G) < \gamma'(G)$, if there are any, will be found among graphs with no isolated vertices <u>not</u> having Property DI. We may as well look among connected graphs, because G has Property DI if and only if each component of G does, and if $\alpha(G) < \gamma'(G)$, then $\alpha(H) < \gamma'(H)$ for some component H of G.

We give our main results on these matters in the next section, while postponing their proofs and our intermediate results to section 3.

2. Main Results

Theorem 2.1. *Every tree of order >1 has Property DI.*

Corollary 2.2. *If F is a forest with no isolated vertices, then $\gamma'(F) \leq \alpha(F)$.*

After proving Theorem 2.1 we discovered that it solves a problem posed in [3], and that the problem is also solved in [4], with a similar but more economical proof. We were going to give our lengthier proof, because it was algorithmic and because we thought it threw light on a conjecture that appeared in an earlier version of this paper, but then along came [5], which settled our conjecture with a very elegant algorithmic proof! So we will not bother with a proof of Theorem 2.1. It follows from the main result in [5], which is the following.

Theorem HLR [5] *If T is a tree of order > 1, and D is a minimum dominating set in T containing at most one leaf, then there is an independent dominating set for T contained in $V(T) \setminus D$.*

Theorem 2.3. *If $\gamma(G) \leq 2$, and G has no isolated vertices then G has Property DI unless $G = K_{m,n}$ for some $m, n > 2$.*

Corollary 2.4. *If $\gamma(G) \leq 2$ and G has no isolated vertices then $\gamma'(G) \leq \alpha(G)$.*

Corollary 2.5. *Suppose that $\gamma(G) = 2$ and G has no isolated vertices. Then the following are equivalent:*

(a) $G = K_{m,n}$ for some $m,n > 2$.

(b) G does not have Property DI.

(c) G does not have Property DI and for each $e \in E(G)$, $G - e$ does have Property DI.

(d) G does not have Property DI and for each $e \in E(\bar{G})$, $G \cup e$ does have Property DI.

Theorem 2.6. *If G has no isolated vertices and $\gamma(G) \leq 4$ then $\gamma'(G) \leq \alpha(G)$.*

3. Proofs and intermediate results

Lemma 3.1. *If $D \subseteq V$ and I is maximal among the independent subsets of $V \setminus D$, then I is dominating in G if and only if $D \subseteq N(I)$.*

Proof. Because I is maximal among the independent subsets of $V \setminus D$, $V \setminus D \subseteq N[I]$. Therefore $V = N[I]$ if and only if each vertex of D is adjacent to some vertex of I. □

Corollary 3.1. *If G has a minimum dominating set D such that there is an independent set $I \subseteq V \setminus D$ with $D \subseteq N(I)$, then G has Property DI.*

Proof. Let $\tilde{I}$ be an independent set such that $I \subseteq \tilde{I} \subseteq V \setminus D$, and $\tilde{I}$ is maximal among all independent subsets of $V \setminus D$. Then $D \subseteq N(I) \subseteq N(\tilde{I})$ so $\tilde{I}$ is a dominating independent set in the complement of D. □

Proof of Theorem 2.3. If $\gamma(G) = 1$, let $D = \{v\}$ be any minimum dominating set. By Lemma 3.1 any independent set I maximal among independent subsets of $V \setminus D$ is a dominating set, so G has Property DI.

Suppose that $\gamma(G) = 2$. Suppose that G does not have Property DI. Then for any minimum dominating set $D = \{x, y\}$ in G, $N(x) \cap N(y) = \emptyset$, for if $z \in N(x) \cap N(y) \subseteq V \setminus D$ then z is contained in a set I, maximal among the independent subsets of $V \setminus D$, and we would have $x, y \in N(z) \subseteq N[I]$ so G would have Property DI after all, by Lemma 3.1.

Let $N(x, D) = N(x) \setminus D$, $N(y, D) = N(y) \setminus D$, which partition $V \setminus D$, by the observation just above. Both sets are non-empty because $\gamma(G) = 2$ and G has no isolated vertices. By the same argument as above, there cannot exist $u \in N(x, D)$ and $v \in N(y, D)$ which are not adjacent in G; that is, every vertex in $N(x, D)$ is adjacent to every vertex in $N(y, D)$. Therefore, if $u \in N(x, D)$ and $v \in N(y, D)$ then $D' = \{u, v\}$ is a minimum dominating set in G. Applying what has been shown about D to D', we conclude that $N(u, D')$ and $N(v, D')$ are disjoint and that $x \in N(u, D')$ and $y \in N(v, D')$ are adjacent. We also conclude that $N(x, D)$ and $N(y, D)$ are each independent because if, for instance, some $u, w \in N(x, D)$, $u \neq w$, are adjacent, then, taking any $v \in N(y, D)$, we would have that $w \in N(u, D') \cap N(v, D')$, $D' = \{u, v\}$.

Thus G is a complete bipartite graph with bipartition $N(x, D) \cup \{y\}$, $N(y, D) \cup \{x\}$. Say $G \cong K_{m,n}$, $m \leq n$. Then $2 \leq m$ because $\gamma(G) = 2$. If $m = 2$ then G does have Property DI; just take D to consist of the 2 vertices on one side of the bipartition and I to be the other side of the bipartition. Therefore, $m, n > 2$. □

Corollary 2.4 follows from the fact that Property DI implies $\gamma' \leq \alpha$, and that if $G = K_{m,n}$, $m, n \geq 2$, then $\gamma'(G) = \gamma(G) = 2$.

Proof of Corollary 2.5 (a) and (b) are equivalent by Theorem 2.3, and the observation that $K_{m,n}$, $m, n > 2$, does not have Property DI. Clearly (c) or (d) implies (b). If $m, n > 2$ then adding or removing an edge to or from $K_{m,n}$ results in a connected graph with domination number 2 which is not $K_{a,b}$ for any a, b. By Theorem 2.3, the modified graph must have Property DI. Thus (a) implies (c) and (d). □

If $S \subseteq V$ let $< S >$ denote the subgraph of G induced by S. (If necessary, $< S >_G$ will denote the same thing.)

Lemma 3.2. *Suppose that G has no isolated vertices, and that $\gamma'(G) = \alpha(G) + c$ for some $c \geq 1$. Suppose that D is a minimum dominating set in G and $I \subseteq S = V \setminus D$ is an independent set of vertices, maximal among independent subsets of S. Let $D_0 = D \setminus N(I)$, $a = a(D, I) = \alpha(< D_0 >)$, and $b = b(D, I) = min[|S_0|;\ S_0 \subseteq S$ and $D_0 \subseteq N(S_0)]$. Then $a + c \leq b \leq |D_0|$.*

Proof. Note that because $\gamma'(G) > \alpha(G)$, G does not have Property DI and therefore, by Corollary 3.1, D_0 is necessarily nonempty. Also, because G has no isolated vertices, every vertex in D must have a neighbor outside of D, i.e., in S. Therefore b is well defined, and $b \leq |D_0|$.

Let $S_0 \subseteq S$ satisfy $|S_0| = b$ and $D_0 \subseteq N(S)$. Then $I \cap S_0 = \emptyset$, and $I \cup S_0$ dominates G. Therefore $\alpha(G) + c = \gamma'(G) \leq |I \cup S_0| = |I| + |S_0| = |I| + b \leq \alpha(G) - a + b$, using the obvious inequality $|I| + a \leq \alpha(G)$. Rearranging $\alpha(G) + c \leq \alpha(G) - a + b$, we have $a + c \leq b$. □

Corollary 3.2. *In the circumstances of Lemma 3.2,* $|D_0| \geq a + c \geq 1 + c \geq 2$ *and* $< D_0 >$ *must have at least* $c \geq 1$ *edges.*

Proof. The first assertion follows directly from the definitions in and conclusions of Lemma 3.2. The second assertion arises from the fact that adding an edge to a graph can decrease its independence number by at most one, so, since $|D_0| - c \geq a = \alpha(< D_0 >)$, $< D_0 >$ must be obtained from the empty graph on $|D_0|$ vertices by inserting at least c edges. □

Proof of Theorem 2.6 Suppose, on the contrary, that $c = \gamma'(G) - \alpha(G) \geq 1$. First suppose that $\gamma(G) = 3$ and that $D = \{x_1, x_2, x_3\}$ is a minimum dominating set in G. Let $S = V \setminus D$, and let $N_1 = N(x_1) \cap S$, $N_2 = N(x_2) \cap S$, and $N_3 = N(x_3) \cap S$. By Corollary 3.2, every set I maximal among independent subsets of S must be contained in one of N_1, N_2, N_3; otherwise, the corresponding D_0 would have at most one element. From this observation it follows that N_1, N_2, N_3 are disjoint, and that any two vertices not in the same set, among these, must be adjacent. Therefore, taking one vertex from each of N_1, N_2, N_3, we obtain a dominating set of size 3 in $S = V \setminus D$, whence $\gamma'(G) \leq 3 = \gamma(G) \leq \alpha(G)$, contrary to supposition.

Now suppose that $\gamma(G) = 4$ and $D = \{x_1, x_2, x_3, x_4\}$ is a minimum dominating set in G. As above, let $S = V \setminus D$ and $N_i = N(x_i) \setminus D = N(x_i) \cap S$, $i = 1, 2, 3, 4$. The N_i are non-empty because G has no isolated vertices. By Corollary 3.2, each maximal independent subset of S can dominate at most 2 elements of D, and so the same is true of any independent subset of S.

We show that there must exist i, j, $1 \leq i < j \leq 4$, and $u \in N_i$, $v \in N_j$, such that u, v are not adjacent. If not, then for every such i, j, every edge uv, $u \in N_i$, $v \in N_j$, $u \neq v$, is an edge of G. Choosing a representative from each N_i, we obtain a dominating set with no more than 4 elements in $V \setminus D$, whence $\gamma'(G) \leq \gamma(G) \leq \alpha(G)$, contrary to supposition.

So, without loss of generality, suppose that $u \in N_3, v \in N_4$, $u \neq v$, and u and v are not adjacent. Let I be any maximal independent subset of S containing u and v. Then $I \subseteq (N_3 \cup N_4) \setminus (N_1 \cup N_2)$, and the D_0 of Lemma 3.2 is $\{x_1, x_2\}$. By previous results, $u, v \notin N_1 \cup N_2$ and $\{u, v\}$ must dominate $N_1 \cup N_2$. Further, with a, b as in Lemma 3.2, by Lemma 3.2 we have that $2 \leq a + c \leq b \leq |D_0| = 2$. We conclude that $a = c = 1$ and $b = 2$. Therefore $N_1 \cap N_2 = \emptyset$ (because $b = 2$), and $x_1x_2 \in E(G)$ (because $a = 1$).

Now, if $y \in N_1$ and $z \in N_2$ were not adjacent, then by the reasoning applied to u and v, y, z must dominate $N_3 \cup N_4$, and so u, v, y, z would be a dominating set in G in $S = V \setminus D$, whence $\gamma' = \gamma \leq \alpha$ again contrary to supposition. So every edge yz, $y \in N_1$, $z \in N_2$, is in G. Choose any $y \in N_1$. Then $D' = \{x_1, y, x_3, x_4\}$ is a minimum dominating set, because $x_1x_2 \in E(G)$ and y dominates N_2. But then u, v, x_2 is an independent set in

$S' = V \backslash D'$ which dominates 3 elements of D', namely x_1, x_3, and x_4, which is impossible by Corollary 3.2 if $\alpha(G) < \gamma'(G)$. [To avoid invoking the corollary, we could as well note that $\{u, v, x_2\}$ dominates all of D', since $\{u, v\}$ dominates $N_1 \cup N_2$, and, therefore $\{y\}$. It follows that G does have Property DI, and so $\gamma'(G) \leq \alpha(G)$, contrary to supposition.] □

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