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A note on the total outer-connected domination number of a tree

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Abstract

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a total outer-connected dominating set of G if D is dominating and $G[V - D]$ is connected. The total outer-connected domination number of G, denoted $\gamma_{tc}(G)$, is the smallest cardinality of a total outer-connected dominating set of G. It is known that if T is a tree of order $n \geq 2$, then $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$. We will provide a constructive characterization for trees achieving the latter bound.

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1. Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $D \subseteq V$ is a *dominating set*, denoted **DS**, of G if every vertex not in $V - D$ is adjacent to a vertex in D. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a **DS**. The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [2, 3].

A set $D \subseteq V$ is a *total dominating set*, denoted **TDS**, if every vertex is adjacent to a vertex in D. Every graph without isolated vertices has a total dominating set, since $D = V$ is such a set. The *total domination number* of G, denoted $\gamma_t(G)$, is the minimum cardinality of a **TDS** of G. A $\gamma_t(G)$*-set* of G is a **TDS** of cardinality $\gamma_t(G)$.

A set of vertices $D \subseteq V$ is a *total outer-connected dominating set*, denoted **TOCDS**, in G if D is a **TDS** and $G[V - D]$ is connected. The *total outer-connected domination number* of G, denoted by $\gamma_{tc}(G)$, is the smallest cardinality of a **TOCDS** of G. A $\gamma_{tc}(G)$-*set* of G is a **TOCDS** of cardinality $\gamma_{tc}(G)$. The notion of total outer-connected domination in graphs was introduced by Cyman in [1].

A *leaf* is a vertex of degree one, while a vertex of degree at least two which is adjacent to a leaf is a *support vertex*. The set of leaves of a graph G will be denoted $L(G)$, while the set of support vertices of G will be denoted $S(G)$.

2. Known results and aim

In [1] Cyman proved the following:

Theorem 2.1. *If T is a tree of order $n \geq 2$, then $\gamma_{tc}(T) \geq \lceil \frac{2n}{3} \rceil$.*

Furthermore, the extremal trees T of order n achieving this lower bound were also constructively characterized.

Let $\mathcal{T}_n = \{T \,|\, T$ is a tree of order n such that $\gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil\}$ and $\mathcal{T} = \cup_{n \geq 3} \mathcal{T}_n$. An operation of **type (a)** on a tree T is defined as attaching a leaf of a P_3 at a vertex v, where v is a vertex of T not belonging to some minimum total outer-connected dominating set of T, and a **type (b)** operation as attaching a P_1 at v, where v belongs to some minimum total outer-connected dominating set of T. Define $\mathcal{C}_n = \{T \,|\, T$ is a tree of order n which can be obtained from the path P_3 by a finite sequence of operations of **type (a)** and **(b)**, where the operation of **type (b)** appears in the sequence exactly $n(mod\ 3)$ times$\}$.

Theorem 2.2. [1] *For $n \geq 3$, $\mathcal{T}_n = \mathcal{C}_n$.*

Note that the operations used in the description of the constructive characterization are formulated in terms of a minimum **TOCDS**. The aim of this paper is to provide a simpler characterization of the trees attaining the bound of Theorem 2.1, where the operations are described independently of a minimum **TOCDS**.

To state our characterization, we define the following operations and classes of trees.

We define a **type (a)** operation on a tree T as attaching a leaf of a P_3 to a vertex v, where v is neither a leaf nor a support vertex of T and has a neighbor w which is a support vertex of degree two; a **type (b)** operation on a tree T is defined as attaching a P_1 to a vertex v, where v is a leaf or a support vertex; while a **type (c)** operation on a tree T is defined as attaching a leaf of a P_2 to any vertex of T.

Let $\mathcal{B}_0 = \{T \,|\, T$ is a tree obtained from the path P_6 by a finite sequence of operations of **type (a)**$\} \cup \{P_3\}$; let $\mathcal{B}_1 = \{T \,|\, T$ is a tree obtained from a tree $T' \in \mathcal{B}_0$ by applying a **type (b)** operation to $T'\}$, and, lastly, let $\mathcal{B}_2 = \{T \,|\, T$ is a tree which can be obtained from a tree $T' \in \mathcal{B}_0$ by applying a **type (c)** operation to T' or applying a **type (b)** operation to T' exactly twice$\}$.

We shall show:

Theorem 2.3. *For* $n \geq 3$, $\mathcal{T} = \cup_{i=0}^{2} \mathcal{B}_i$.

3. Proof of Theorem 2.3

Before proceeding with the proof of Theorem 2.3, we observe the following:

Observation 3.1. *If* $T \in \mathcal{B}_0 - \{P_3\}$ *has order* $n \geq 6$, *then* $D(T) = L(T) \cup S(T)$ *is a* **TOCDS** *of* T *of cardinality* $\frac{2n}{3}$.

Proof. We use induction on k, the number of steps required to construct T. If $k = 0$, then $T \cong P_6$, and D is a **TOCDS** of cardinality $4 = \frac{2\times 6}{3}$. So suppose T can be constructed from P_6 by a sequence of $k \geq 1$ applications of operation **type (a)**, and suppose that if T' is a tree which can be obtained from P_6 by $k-1$ applications of operation **type (a)**, then $D(T')$ is a **TOCDS** of cardinality $\frac{2n(T')}{3}$. As $T \in \mathcal{B}_0 - \{P_3\}$, T can be obtained from a tree T' by applying a **type (a)** operation. Suppose $\{x_1, x_2, x_3\}$ is the vertex set of the P_3 whose leaf x_3 is joined to vertex $v \in V(T') - D(T')$. By the induction assumption, $D(T') = L(T') \cup S(T')$ is a **TOCDS** of cardinality $\frac{2n(T')}{3}$. As $v \notin D(T')$, the set $D(T') \cup \{x_1, x_2\} = D(T)$ is a **TOCDS** of cardinality $\frac{2n(T)}{3}$. □

We are now ready to begin with the proof of Theorem 2.3.

Proof. We first show that $\cup_{i=0}^{2} \mathcal{B}_i \subseteq \mathcal{T}$.

Suppose $T \in \mathcal{B}_0$. If $T \cong P_3$, then $\gamma_{tc}(T) = 2 = \lceil \frac{2n(T)}{3} \rceil$. If T is not a P_3, then, by Observation 3.1 and Theorem 2.1, $\gamma_{tc}(T) = \lceil \frac{2n(T)}{3} \rceil$, and so $T \in \mathcal{T}$.

Suppose $T \in \mathcal{B}_1$. Then $n = n(T) = 3k + 1$ for some positive integer k, and T can be constructed from $T' \in \mathcal{B}_0$ by applying a **type (b)** operation to T'. Let u be the P_1 attached. By Observation 3.1, $D(T')$ is a **TOCDS** of cardinality $2k$. But then $D \cup \{u\}$ is a **TOCDS** of T of cardinality $2k + 1$. By Theorem 2.1, $\gamma_{tc}(T) \geq \lceil \frac{2(3k+1)}{3} \rceil = 2k + 1$, and so $\gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil$, whence $T \in \mathcal{T}$.

Suppose $T \in \mathcal{B}_2$. Then $n = n(T) = 3k + 2$ for some positive integer k, and T can be constructed from $T' \in \mathcal{B}_0$ by applying a **type (c)** operation to T' or applying a **type (b)** operation exactly twice to T'. Let u and u' be the added vertices. By Observation 3.1, $D(T')$ is a **TOCDS** of cardinality $2k$. But then $D \cup \{u, u'\}$ is a **TOCDS** of T of cardinality $2k+2$. By Theorem 2.1, $\gamma_{tc}(T) \geq \lceil \frac{2(3k+2)}{3} \rceil = 2k+2$, and so $\gamma_{tc}(T) = \lceil \frac{2n}{3} \rceil$, whence $T \in \mathcal{T}$.

We show next that $\mathcal{T} \subseteq \cup_{i=0}^{2} \mathcal{B}_i$ by employing induction on the order n of the tree $T \in \mathcal{T}$. If $n = 3$, then $T \cong P_3 \in \mathcal{B}_0$. Suppose $T \in \mathcal{T}$ has order $n = n(T) \geq 1$, and assume that if T' is a tree with $3 \leq n(T') < n$ and $T' \in \mathcal{T}$, then $T' \in \cup_{i=0}^{2} \mathcal{B}_i$. Let D be any $\gamma_{tc}(T)-$set. We first make a few observations:

Observation 3.2. *Let* $T \in \mathcal{T}$ *and suppose* v_1 *is a leaf of* T. *If* $D - \{v_1\}$ *is a* ***TOCDS*** *of* $T - v_1$, *then* $n \not\equiv 0 (mod\ 3)$ *and* $T - v_1 \in \mathcal{T}$.

Proof. Suppose that v_1 is a leaf of T, and that $D - \{v_1\}$ is a **TOCDS** of $T - v_1$. Note that $\left\lceil \frac{2(n-1)}{3} \right\rceil \le \gamma_{tc}(T - v_1) \le \lceil \frac{2n}{3} \rceil - 1$.

If $n = 3k$ for some positive integer $k \ge 1$, then we obtain $\left\lceil \frac{2(3k-1)}{3} \right\rceil \le \left\lceil \frac{2(3k)}{3} \right\rceil - 1$, and so $2k \le 2k - 1$, which is impossible. Thus, $n = 3k + 1$ or $n = 3k + 2$ for some positive integer $k \ge 1$.

If $n = 3k+1$, then $\left\lceil \frac{2(3k)}{3} \right\rceil \le \gamma_{tc}(T - v_1) \le \left\lceil \frac{2(3k+1)}{3} \right\rceil - 1$, which implies that $2k \le \gamma_{tc}(T - v_1) \le 2k$ and so $T - \{v_1\} \in \mathcal{T}$. If $n = 3k+2$, then $\left\lceil \frac{2(3k+1)}{3} \right\rceil \le \gamma_{tc}(T - v_1) \le \left\lceil \frac{2(3k+2)}{3} \right\rceil - 1$, which implies that $2k + 1 \le \gamma_{tc}(T - v_1) \le 2k + 1$ and so $T - v_1 \in \mathcal{T}$. □

Observation 3.3. *Let $T \in \mathcal{T}$ and suppose v_1 and v_2 are either both leaves of T or v_1 is a degree two support vertex of T adjacent to a leaf v_2. If $D - \{v_1, v_2\}$ is a **TOCDS** of $T - v_1 - v_2$, then $T - v_1 - v_2 \in \mathcal{T}$ and $n \equiv 2 (mod\ 3)$.*

Proof. Suppose that v_1 and v_2 are either both leaves of T or v_1 is a degree two support vertex of T adjacent to a leaf v_2 and that $D - \{v_1, v_2\}$ is a **TOCDS** of $T - v_1 - v_2$. Note that $\left\lceil \frac{2(n-2)}{3} \right\rceil \le \gamma_{tc}(T - v_1 - v_2) \le \lceil \frac{2n}{3} \rceil - 2$.

Suppose that either $n = 3k$ or $n = 3k + 1$ for some positive integer k. If $n = 3k$, then $\left\lceil \frac{2(3k-2)}{3} \right\rceil \le \left\lceil \frac{2(3k)}{3} \right\rceil - 2$, and so $2k - 1 \le 2k - 2$, which is impossible. If $n = 3k + 1$, then $\lceil \frac{2(3k-1)}{3} \rceil \le \left\lceil \frac{2(3k+1)}{3} \right\rceil - 2$, and so $2k \le 2k - 1$, which is impossible.

Thus, $n = 3k + 2$. Hence, $n = 3k + 2$ for some positive integer k, and so $\left\lceil \frac{2(3k)}{3} \right\rceil \le \left\lceil \frac{2(3k+2)}{3} \right\rceil - 2$, whence $2k \le \gamma_{tc}(T - v_1 - v_2) \le 2k$ and so $T - v_1 - v_2 \in \mathcal{T}$. □

Let $v \in V - D$. Furthermore, since D is a **DS**, vertex v has a neighbor $u_0 \in D$. Consider a path $u_0, u_1, \ldots, u_t$ where $u_i \in D$ for all $0 \le i \le t$. Such a path exists since D is total. If t is a maximum and u_t is a leaf, then we call such a path a $P(u_0, u_t)$-path.

Observation 3.4. *No vertex u_i, $i \ne 0$, in $u_0, u_1, \ldots, u_t$ is adjacent to a vertex of $V - D$. Also, u_0 is adjacent to only v in $V - D$.*

Proof. Suppose u_i, for $i \ne 0$, is adjacent to a vertex $u' \in V - D$. Since $\langle V - D \rangle$ is connected, there is a path in $\langle V - D \rangle$ from u' to v and so T contains a cycle, which is a contradiction. If u_0 is adjacent to a vertex u' other than v in $V - D$ then, once again, T contains a cycle, which is a contradiction. □

Observation 3.5. *For any $P(u_0, u_t)$-path we may assume that $t = 1$. Furthermore,* $\deg(u_0) = 2$.

Proof. Suppose, to the contrary, that $t \geq 2$. By Observation 3.2 and our induction assumption, it follows that the tree $T - u_t$ is either in $\mathcal{B}_0$ or in $\mathcal{B}_1$. If $T - u_t \in \mathcal{B}_0$, then $T \in \mathcal{B}_1$. Similarly, if $T - u_t \in \mathcal{B}_1$, then $T \in \mathcal{B}_2$.

Hence, $t = 1$. Suppose $\deg(u_0) \geq 3$. Then, by Observation 3.4, vertex u_0 is a support vertex, where every vertex in $N(u_0) - \{v\}$ is a leaf. By Observation 3.2 and our induction assumption, it follows by the same argument as in the previous paragraph that $T \in \mathcal{B}_1 \cup \mathcal{B}_2$. Hence, we may assume that $\deg(u_0) = 2$. □

Observation 3.6. *We may assume that* $N(v) \cap D = \{u_0\}$.

Proof. Suppose, to the contrary, that v is adjacent to some vertex v_1 in D other than u_0. By Observation 3.5 we have that v_1 is a degree two support vertex adjacent to a leaf v_2. Note that $D - \{v_1, v_2\}$ is a **TOCDS** of $T - v_1 - v_2$. By Observation 3.3 and our induction assumption, it follows that the tree $T - v_1 - v_2$ is in $\mathcal{B}_1$. Hence, $T \in \mathcal{B}_2$. □

By Observation 3.5 and 3.6, each vertex in $V - D$ is adjacent to exactly one vertex in D. Furthermore, every vertex in D is either a degree two support vertex adjacent to exactly one vertex in $V - D$, or a leaf adjacent to a degree two support vertex that is adjacent to exactly one vertex in $V - D$.

Hence, $n \equiv 0 (mod\ 3)$ and so $n = 3k$ for a positive integer $k \geq 2$.

Let v be the endpoint of a diametrical path $v = v_1, \ldots, v_\ell$ in $\langle V - D \rangle$. As T contains no cycles, v is a leaf of $\langle V - D \rangle$. Thus, $\deg(v) = 2$. By Observation 3.6, $|V - D| \geq 2$, and so v_2 is adjacent to a degree two support vertex. If $|V - D| = 2$, then $T \cong P_6 \in \mathcal{B}_0$. We therefore assume $|V - D| \geq 3$. Note that v_2 is neither a leaf nor a support vertex of $T' = T - v_1 - u_0 - u_1$. Moreover, $D - \{u_0, u_1\}$ is a **TOCDS** of T', and so $\left\lceil \frac{2(3k-3)}{3} \right\rceil \leq \gamma_{tc}(T') \leq \left\lceil \frac{2(3k)}{3} \right\rceil - 2$, which implies that $2k - 2 \leq \gamma_{tc}(T') \leq 2k - 2$ and so $T' \in \mathcal{T}$. By our induction assumption $T' \in \mathcal{B}_0$. Hence, $T \in \mathcal{B}_0$. □

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