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Minus edge k-subdomination numbers in graphs

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Abstract

The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 0, 1\}$. If $\sum_{x\in N[e]} f(x) \geq 1$ for at least k edges e of G, then f is called a minus edge k-subdominating function of G. The minimum of the values $\sum_{e\in E(G)} f(e)$, taken over all minus edge k-subdominating functions f of G, is called the minus edge k-subdomination number of G and is denoted by $\gamma'_{km}(G)$. In this note we initiate the study of minus edge k-subdomination in graphs and present some (sharp) bounds for this parameter.

Keywords: minus edge dominating function; minus edge domination number; minus edge k-subdominating function; minus edge k-subdomination number.

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1. Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [8] for terminology and notation which are not defined here. The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G. It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 0, 1\}$

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and a subset S of $E(G)$ we define $f(S)=\sum_{e\in S} f(e)$. If $S=N_G[e]$ for some $e\in E$, then we denote $f(S)$ by $f[e]$. For each vertex $v\in V(G)$ we also define $f(v)=\sum_{e\in E(v)} f(e)$, where $E(v)$ is the set of all edges at vertex v. A function $f:E(G)\longrightarrow\{-1,0,1\}$ is called a *minus edge k-subdominating function* (MEkSDF) of G, if $f[e]\geq 1$ for at least k edges e of G. The minimum of the values $f(E(G))$, taken over all minus edge k-subdominating functions f of G, is called the *minus edge k-subdomination number* of G and is denoted by $\gamma'_{km}(G)$. The minus edge k-subdominating function f of G with $f(E(G))=\gamma'_{km}(G)$ is called *$\gamma'_{km}(G)$-function*. For any minus edge k-subdominating function f of G we define $P=\{e\in E(G)\mid f(e)=1\}$, $M=\{e\in E(G)\mid f(e)=-1\}$, $Z=\{e\in E(G)\mid f(e)=0\}$ and $X=\{e\in E(G)\mid f[e]\geq 1\}$.

If $k=m$, then the minus edge k-subdomination number is called the *minus edge domination number*. The minus edge domination number was introduced by Xu and Zhou in [9] and denoted by $\gamma'_m(G)$.

A function $f:E(G)\longrightarrow\{-1,1\}$ is called a *signed edge k-subdominating function* (SEkSDF) of G, if $f[e]\geq 1$ for at least k edges e of G. The minimum of the values $f(E(G))$, taken over all signed edge k-subdominating functions f of G, is called the *signed edge k-subdomination number* of G and is denoted by $\gamma'_{ks}(G)$. The signed edge k-subdominating number was introduced by Khodkar et al. in [6]. Since every signed edge k-subdominating function of G is a minus edge k-subdominating function for G, we have

$$\gamma'_{ks}(G)\geq\gamma'_{km}(G). \tag{1}$$

A *minus k-subdominating function* (MkSF) for G is defined in [1] as a function $f:V(G)\longrightarrow\{-1,0,1\}$ such that $f(N[v])\geq 1$ for at least k vertices of G where $N[v]$ is the closed neighborhood of v. The *minus k-subdomination number* of a graph G, denoted by $\gamma^{-101}_{ks}(G)$, is equal to $\min\{f(V(G))\mid f \text{ is a minus } MkSF \text{ of } G\}$. The minus k-subdomination number has been studied by several authors (see for example [3, 4, 5]).

If $k=m$, then the minus k-subdomination number is called the *minus domination number*. The minus domination number was introduced by Dunbar et al. in [2].

In this note we initiate the study of minus edge k-subdomination in graphs and present some (sharp) bounds for this parameter.

Here are some well-known results on $\gamma'_m(G)$, $\gamma'_{ks}(G)$ and $\gamma^{-101}_{ks}(G)$.

Theorem A. [7] *Let G be a connected graph of order $n\geq 2$ and size m. Then*

$$\gamma'_m(G)\geq n-m.$$

Theorem B. [9] *For any connected graph G of order $n\geq 2$, $\gamma'_m(G)\geq\frac{(4-n)\Delta}{4}$.*

Theorem C. [9] *For any connected graph G of order $n\geq 2$ and size m,*

$$\gamma'_m(G)\geq\frac{4m-(\Delta-\delta)n^2}{4(2\Delta-1)}$$

Theorem D. [6] *Let G be a connected graph of order $n \geq 3$, size m and $1 \leq k \leq m-1$. Then*

$$\gamma'_{ks}(G) \geq n + k + 1 - 2m.$$

Theorem E. [6] *Let $\psi(m) = \min\{\gamma'_s(G) \mid G \text{ is a graph of size } m\}$. Then for any simple graph G of order $n \geq 3$, size m and integer $1 \leq k \leq m$,*

$$\gamma'_{ks}(G) \geq \Psi(t) - (m - t),$$

for some integer $k \leq t \leq m$.

Theorem F. [2]

1. *For the path P_n, $\gamma_{ns}^{-101}(P_n) = \lceil \frac{n}{3} \rceil$.*
2. *If $n \geq 3$, then $\gamma_{ns}^{-101}(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem G. [1] *For $n \geq 2$ and $1 \leq k \leq n-1$, $\gamma_{ks}^{-101}(P_n) = \lceil \frac{k}{3} \rceil + k - n + 1$.*

Theorem H. [4] *If $n \geq 3$ and $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-101}(C_n) = \begin{cases} \lceil \frac{n-2}{3} \rceil & \text{if } k = n-1 \text{ and } k \equiv 0,1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

The proof of the following theorem is straightforward and therefore omitted.

Theorem 1.1. *For any graph G of order $n \geq 2$ which has no isolates,*

$$\gamma'_{km}(G) = \gamma_{ks}^{-101}(L(G)).$$

Theorems 1.1, F, G and H lead to:

Corollary 1.2. *For $n \geq 2$ and $1 \leq k \leq n$,*

$$\gamma'_{km}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } k = n \\ \lceil \frac{k}{3} \rceil + k - n + 1 & \text{otherwise.} \end{cases}$$

Corollary 1.3. *For $n \geq 3$ and $1 \leq k \leq n$,*

$$\gamma'_{km}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } k = n \\ \lceil \frac{n-2}{3} \rceil & \text{if } k = n-1 \text{ and } k \equiv 0,1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

2. Lower bounds on the MEkSDNs of graphs

Let f be an MEkSDF of G. An edge e is said to be a $+1$ edge if $f(e) = 1$, a 0 edge if $f(e) = 0$ and it is said to be a -1 edge if $f(e) = -1$. In this section we first present a lower bound for $\gamma'_{km}(G)$ in terms of k, the size of G, the minimum degree and the maximum degree of G and then we find a lower bound for $\gamma'_{km}(G)$ in terms of k, the order and the size of G. Finally, we generalize Theorem E to the minus edge k-subdomination number.

Theorem 2.1. *Let G be a simple graph of size m, minimum degree δ, maximum degree Δ and no isolates. Then*

$$\gamma'_{km}(G) \geq \frac{2k\delta}{2\Delta - 1} - m.$$

Proof. Let $(d_1, \ldots, d_n)$ be the degree sequence of G where $d_1 \leq d_2 \leq \ldots \leq d_n$. Assume g is a $\gamma'_{km}(G)$-function of G and let $g[e] \geq 1$ for k distinct edges e in $\{e_{j_1} = u_{j_1}v_{j_1}, \ldots, e_{j_k} = u_{j_k}v_{j_k}\}$. Define $f : E(G) \longrightarrow \{0, \frac{1}{2}, 1\}$ by $f(e) = \frac{g(e)+1}{2}$ for each $e \in E(G)$. We have

$$\begin{aligned} \textstyle\sum_{i=1}^{k} f(N_G[e_{j_i}]) &\geq \textstyle\sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) + \deg(u_{j_i}) + \deg(v_{j_i}) - 1}{2} \\ &\geq k\delta + \textstyle\sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) - 1}{2} \\ &\geq k\delta + \textstyle\sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) - 1}{2} \\ &\geq k\delta. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} \textstyle\sum_{i=1}^{k} f(N_G[e_{j_i}]) &\leq \textstyle\sum_{e\in E} f(N_G[e]) = \sum_{e=uv\in E}(\deg(u) + \deg(v) - 1)f(e) \\ &\leq \textstyle\sum_{e\in E}(2\Delta - 1)f(e) \\ &= (2\Delta - 1)f(E(G)). \end{aligned} \tag{3}$$

By (1) and (2), $f(E(G)) \geq \dfrac{k\delta}{2\Delta - 1}$. Since $g(E(G)) = 2f(E(G)) - m$,

$$\gamma'_{km}(G) = g(E(G)) \geq \frac{2k\delta}{2\Delta - 1} - m,$$

as desired. □

As an immediate consequence of Theorem 2.1 we have:

Corollary 2.2. *For every r-regular $(r \geq 1)$ graph G of size m, $\gamma'_{km}(G) \geq \dfrac{2rk}{2r - 1} - m$. Furthermore, this bound is sharp when $r = 1$.*

Now we prove that for any simple connected graph G of size $m \geq 2$ and any integer $1 \leq k \leq m-1$, $\gamma'_{km}(G) \geq n+k+1-2m$.

Theorem 2.3. *Let G be a simple connected graph of order $n \geq 3$, size m and $1 \leq k \leq m-1$. Then*

$$\gamma'_{km}(G) \geq n+k+1-2m.$$

Furthermore, the bound is sharp for each odd integer $k \geq 7$.

Proof. The proof is by induction on m. Obviously, the statement is true for $m = 2, 3$. Assume the statement is true for all simple connected graphs of size less than m, where $m \geq 4$. Let G be a simple connected graph of size m and let f be a $\gamma'_{km}(G)$-function. We may assume $Z \neq \emptyset$, for otherwise we have $\gamma'_{km}(G) = \gamma'_{ks}(G)$ and the result follows by Theorem D. We consider two cases.

Case 1. There exists a pendant edge $e = uv \in E(G)$ for which $f(e) = 0$.

Let $\deg(u) = 1$ and $G' = G-u$. First let $e \notin X$. If $k \leq m-2$, then obviously f, restricted to G', is an MEkSDF of G' and by the inductive hypothesis, we have

$$f(E(G)) = f(E(G')) \geq (n-1)+k+1-2(m-1) = n+k+2-2m.$$

If $k = m-1$, then by Theorem A we have

$$f(E(G)) = f(E(G')) \geq (n-1)-(m-1) = n+k+1-2m.$$

Now let $e \in X$. If $k = 1$, then f must assign $+1$ to an edge incident to v and so

$$f(E(G)) \geq 3-m \geq n+k+1-2m.$$

We therefore assume $k \geq 2$. Then f, restricted to G', is an ME(k-1)SDF of G', and by the inductive hypothesis we have

$$\gamma'_{km}(G) = f(E(G)) = f(E(G')) \geq (n-1)+(k-1)+1-2(m-1) \geq n+k+1-2m.$$

Case 2. For any pendant edge e in G, $f(e) \neq 0$.

Since $Z \neq \emptyset$, there exists a non-pendant edge $e = uv \in E(G)$ for which $f(e) = 0$. First let e be a non-bridge edge. If $e \notin X$, then f, restricted to $G-e$, is an MEkSDF of $G-e$. If $k \leq m-2$, then by the inductive hypothesis

$$\gamma'_{km}(G) = f(E(G)) = f(E(G-e)) \geq n+k+1-2(m-1) = n+k+3-2m.$$

If $k = m-1$, then f, restricted to $G-e$, is an MEDF of $G-e$ and by Theorem A we have

$$\gamma'_{km}(G) = f(E(G)) = f(E(G-e)) \geq n-(m-1) = n-m+1 = n+k+2-2m.$$

Let $e \in X$. If $k = 1$, then an argument similar to that described in Case 1 shows that $\gamma'_{km}(G) \geq n + k + 1 - 2m$. Assume that $k \geq 2$. Then f, restricted to $G - e$, is an ME$(k-1)$SDF of $G - e$, by the inductive hypothesis,

$$\gamma'_{km}(G) = f(E(G)) = f(E(G-e)) \geq n + (k-1) + 1 - 2(m-1) = n + k + 2 - 2m.$$

Now assume e is a bridge and G_1 and G_2 are the connected components of $G - e$ and $u \in G_1$. We consider two subcases.

Subcase 2.1 For $i = 1, 2$, $X \cap E(G_i) \neq \emptyset$. Let $|X \cap E(G_1)| = k_1$ and $|X \cap E(G_2)| = k_2$. Then for $i = 1, 2$, the function f, restricted to G_i, is an MEk_iSDF for G_i. Hence, $\gamma'_{k_i m}(G_i) \leq f(E(G_i))$ for $i = 1, 2$. First let $E(G_i) \subseteq X$ for $i = 1, 2$. By Theorem A,

$$\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G_2)) \geq n - (m-1) \geq n + k + 2 - 2m.$$

Now without loss of generality we assume $E(G_1) \nsubseteq X$. If $E(G_2) \subseteq X$, then by the inductive hypothesis and Theorem A

$$\begin{aligned}
\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G_2)) \\
&\geq |V(G_1)| + k_1 + 1 - 2|E(G_1)| + |V(G_2)| - |E(G_2)| \\
&= n + k_1 + 1 - 2|E(G_1)| - |E(G_2)| + k_2 - |E(G_2)| \\
&\geq n + (k-1) + 1 - 2(m-1) = n + k + 2 - 2m.
\end{aligned}$$

If $E(G_2) \nsubseteq X$, then by inductive hypothesis

$$\begin{aligned}
\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G_2)) \\
&\geq |V(G_1)| + k_1 + 1 - 2|E(G_1)| + |V(G_2)| + k_2 + 1 - 2|E(G_2)| \\
&\geq n + k + 3 - 2m.
\end{aligned}$$

Subcase 2.2 $X \cap E(G_1) = \emptyset$ (the case $X \cap E(G_2) = \emptyset$ is similar). First let $e \notin X$ and $G' = G_2 + uv$. We claim that f assigns -1 to all edges of G_1. If $E(G_1) \cap P \neq \emptyset$, where $P = \{e \in E(G) \mid f(e) = 1\}$, then we define $g : E(G) \longrightarrow \{-1, 0, +1\}$ by $g(e) = -1$ if $e \in E(G_1)$ and $g(e) = f(e)$ if $e \in E(G) \setminus E(G_1)$. Then g is a MEkSDF of G of weight less than f, a contradiction. This proves our claim. Then $k \leq k_2 < |E(G')|$ and by the inductive hypothesis we have

$$\begin{aligned}
\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G')) \\
&\geq -|E(G_1)| + |V(G')| + k_2 + 1 - 2|E(G')| \\
&\geq -2m + (n+1) + k_2 + 1 + (|E(G_1)| - |V(G_1)|) \\
&\geq n + k_2 + 1 - 2m \\
&\geq n + k + 1 - 2m.
\end{aligned}$$

Now assume $e \in X$. We may assume $k \geq 2$ for otherwise the result follows as in Case 1. If $E(G') \nsubseteq X$ and $f(v) \geq 1$, then by inductive hypothesis we have

$$\begin{aligned}
\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G')) \\
&\geq -|E(G_1)| + |V(G')| + k + 1 - 2|E(G')| \\
&= -2m + (n+1) + k + 1 + (|E(G_1)| - |V(G_1)|) \\
&\geq n + k + 1 - 2m.
\end{aligned}$$

If $E(G') \not\subseteq X$ and $f(v) \leq 0$, then f restricted to G' is an ME$(k-1)$SDF of G' and by inductive hypothesis we have

$$\begin{aligned}\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G')) \\ &\geq (2 - |E(G_1)|) + |V(G')| + (k-1) + 1 - 2|E(G')| \\ &= -2m + (n+1) + k + 2 + (|E(G_1)| - |V(G_1)|) \\ &\geq n + k + 2 - 2m.\end{aligned}$$

If $E(G') \subseteq X$ and $f(v) \geq 1$, then f restricted to G' is an MEDF of G' and by Theorem A we have

$$\begin{aligned}\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G')) \\ &\geq -|E(G_1)| + |V(G')| - |E(G')| \\ &= n + k + 1 - 2m + (m - k - |V(G_1)|) \\ &\geq n + k + 2 - 2m.\end{aligned}$$

Finally, if $E(G') \subseteq X$ and $f(v) \leq 0$, then f restricted to G' is an ME$(k-1)$SDF of G' and by inductive hypothesis we have

$$\begin{aligned}\gamma'_{km}(G) &= f(E(G)) = f(E(G_1)) + f(E(G')) \\ &\geq -|E(G_1)| + |V(G')| + (k-1) + 1 - 2|E(G')| \\ &= n + k + 1 - 2m + (|E(G_1)| - |V(G_1)|) \\ &\geq n + k + 1 - 2m.\end{aligned}$$

To prove sharpness, we consider two cases.

- $k \geq 7$ is odd and $k = m - 1$. Let G be obtained from star $K_{1,k-3}$ with vertex set$\{v, v_1, \ldots, v_{k-3}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-3\}$ by adding three pendant edges $v_1v'_1, v_2v'_2, v_3v'_3$ and an edge v_1v_2. Define $f : V(G) \longrightarrow \{-1, 0, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an MEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.
- $k \geq 7$ is odd and $k \leq m - 2$. Let G be obtained from star $K_{1,k-2}$ with vertex set $\{v, v_1, \ldots, v_{k-2}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-2\}$ by adding pendant edges $v_1v'_1$, $v_2v'_2$, $v_3v^j_3$ for $j = 1, 2, \ldots, m - k - 1$ and v_1v_2. Define $f : V(G) \longrightarrow \{-1, 0, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an MEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.

This completes the proof. □

Xu and Zhou in [9] defined $\eta(m) = \min\{\gamma'_m(G) \mid G \text{ is a graph of size } m\}$ for any positive integer m. The proof of the following Lemma is straightforward and therefore omitted.

Lemma 2.4. *Let η be as above. Then*

1. *$m \geq \eta(m)$ for every positive integer m, and*

2. $\eta(a) + \eta(b) \geq \eta(a + b)$ *for each pair of positive integers* a *and* b.

The proof of the following theorem is essentially similar to the proof of Theorem 6 of [6].

Theorem 2.5. *For any simple graph* G *of order* $n \geq 3$ *without isolated vertices, size* m *and integer* $1 \leq k \leq m$,

$$\gamma'_{km}(G) \geq \eta(t) - (m - t),$$

for some integer $k \leq t \leq m$. *Furthermore, this bound is sharp when* $t = k$.

Proof. The statement holds for all simple graphs of size $m = 1, 2, 3$. Now assume $m \geq 4$. Let, to the contrary, G be a simple graph of size $m \geq 4$ such that $\gamma'_{km}(G) < \eta(t) - (m-t)$ for every integer $k \leq t \leq m$. Choose such a graph G with as few edges as possible for which $\lambda(G) + |T(G)|$ is maximum, where $\lambda(G)$ denotes the number of components of G and $T(G) = \{u \in V(G) \mid \deg(u) \leq 2\}$. Let f be a $\gamma'_{km}(G)$-function. We may assume $Z \neq \emptyset$, for otherwise we have $\gamma'_{km}(G) = \gamma'_{ks}(G)$ and the result follows by Theorem E. Let $G_1, \ldots, G_{\lambda(G)}$ be the connected components of G. If $G_i \simeq K_2$ for each $1 \leq i \leq \lambda(G)$, then obviously

$$\gamma'_{km}(G) = k - (m - k) \geq \eta(k) - (m - k).$$

Let G have a component H of size at least 2.

Claim 1. $E(H) \cap (M \cup Z) \subseteq X$.

Let $e \in E(H) \cap M$ (the case $e \in E(H) \cap Z$ is similar). Suppose that, to the contrary, $e \notin X$. Assume G' is obtained from $G - e$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 0, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$. Obviously, g is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$. This contradicts the assumptions on G. Thus $e \in X$.

Claim 2. There is no non-pendant edge $e = uv \in E(H) \cap Z$.

Let $e = uv \in E(H) \cap Z$ be a non-pendant edge. Since $e \in X$, $f(u) \geq 1$ or $f(v) \geq 1$. Let, without loss of generality, $f(u) \geq 1$ and let G' be obtained from $G - e$ by adding a pendant edge uv'. Then obviously $g : E(G') \longrightarrow \{-1, 0, 1\}$, which is defined by $g(uv') = 0$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$, is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$. This contradicts the assumptions on G.

Claim 3. For every non-pendant edge $e = uv \in E(H) \cap M$ we have $\deg(u) = \deg(v) = 2$.

If $f(u) \geq 1$ (the case $f(v) \geq 1$ is similar), then an argument similar to that described in claim 2 leads to a contradiction. Hence, $f(u) = f(v) = 0$. Since e is a non-pendant edge, $\deg(u), \deg(v) \geq 2$. Let $\deg(u) \geq 3$ (the case where $\deg(v) \geq 3$ is similar). Then there is a $+1$ edge $e' = uw$ at u. Assume G' is obtained from $G - \{e, e'\}$ by adding a new vertex z and two new edges vz and wz. Define $g : E(G') \longrightarrow \{-1, 0, 1\}$ by $g(vz) = -1$, $g(wz) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, e'\}$. Obviously, g is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$, a contradiction. Hence, $\deg(u) = \deg(v) = 2$.

Claim 4. Let $e = uv \in E(H) \cap M$ be a non-pendant edge and $uu', vv' \in E(G)$. Then $uu', vv' \in X$.

Let, to the contrary, $uu' \notin X$ (the case $vv' \notin X$ is similar). Since $e \in X$, $f(uu') = f(vv') = 1$. Suppose that $\deg(u') = 1$ and G' is obtained from $G - \{e, uu'\}$ by adding a pendant edge vv_1 and a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 0, 1\}$ by $g(vv_1) = -1$, $g(u_0v_0) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu'\}$. Then g is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$, a contradictions. Therefore $\deg(u') \geq 2$. Similarly, we can see that $\deg(v') \geq 2$.

First let $u' = v'$. Since $uu' \notin X$, we have $vv' \notin X$. Suppose that there exists a -1 or 0 pendant edge $u'z$ at u'. By Claim 1, $u'z \in X$, which implies that $f(u') \geq 1$. Let G' be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Define $g : E(G') \longrightarrow \{-1, 0, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$. Obviously, g is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$, a contradiction. Therefore, there is no -1 or 0 pendant edge at $u' = v'$. If there exists a -1 non-pendant edge at u', then an argument similar to that described in Claim 3 shows that $\deg(u') = 2$, a contradiction. Thus every edge at u' is $+1$ edge. This forces $uu' \in X$, a contradiction.

Now let $u' \neq v'$. Since we have assumed $uu' \notin X$, it follows that $f(u') \leq 1$. If there is a -1 or 0 pendant edge $u'w$ at u', then by Claim 1 we have $u'w \in X$ and hence, $f(u') = f(N[u'w]) \geq 1$. If there is a -1 non-pendant edge at u', then $\deg(u') = 2$ by Claim 3 and hence, $f(u') = 0$. It follows that $f(u') = 0, 1$.

When $f(u') = 1$, define G' to be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0. Then $g : E(G') \longrightarrow \{-1, 0, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$ is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$, a contradiction. Therefore $f(u') = 0$ and hence, there exists a -1 edge $u'u''$ at u'. By claim 1 we have $\deg(u'') \neq 1$. Hence, $\deg(u'') = 2$ (see Claim 3). Let G' be obtained from $G - \{e, uu', u'u''\}$ by adding a new component u_0v_0 and a new vertex z along with two edges $u''z, zv$. Then $g : E(G') \longrightarrow \{-1, 0, 1\}$ defined by $g(u_0v_0) = -1$, $g(u''z) = -1$, $g(zv) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu', u'u''\}$ is an MEkSDF of G' with $g(E(G')) = f(E(G))$ and $\lambda(G') + |T(G')| > \lambda(G) + |T(G)|$, a contradiction. Therefore $uu' \in X$, a contradiction.

Claim 5. $E(H) \cap P \subseteq X$.

Let $e = uv \in E(H) \cap P$. If there is a -1 non-pendant edge at u or at v, then by Claim 4 we have $e \in X$. If there exists a -1 or 0 pendant edge e' at u and no 0 or -1 pendant edge at v, then since $e' \in X$, $f(u) \geq 1$. Since there is not any -1 edge or 0 edge at v, $f(v) \geq 1$. Hence $f(N[uv]) \geq 1$ and $e \in X$. If there exist -1 or 0 pendant edges at u and v then $f(u), f(v) \geq 1$. Thus $e \in X$. Obviously , if there is not any -1 pendant edges at u and v, we see that $e \in X$.

Let $G_1, \ldots, G_s$ be the connected components of G for which $E(G_i) \subseteq X$. Then, the restriction $f|_{G_i}$ is a γ'_m-function on G_i for each $1 \leq i \leq s$. Now by Claims 1 and 5, $X \cap [\cup_{i=s+1}^{\lambda(G)} E(G_i)] = \emptyset$.

Let $|E(G_i)| = m_i$ for each $1 \leq i \leq \lambda(G)$. Then $|X| = \sum_{i=1}^{s} m_i \geq k$ and $\sum_{i=s+1}^{\lambda(G)} m_i \leq m - k$. Then by Lemma 2.4,

$$\begin{array}{rcl} \gamma'_{km}(G) & = & \sum_{i=1}^{s} \gamma'_m(G_i) - \sum_{i=s+1}^{\lambda(G)} m_i \\ & \geq & \sum_{i=1}^{s} \eta(m_i) - \sum_{i=s+1}^{\lambda(G)} m_i \\ & \geq & \eta(\sum_{i=1}^{s} m_i) - \sum_{i=s+1}^{\lambda(G)} m_i \\ & = & \eta(t) - (m - t). \end{array}$$

Where $t = \sum_{i=1}^{s} m_i \geq k$.
In order to prove that the lower bound is sharp when $t = k$, let H_1 be a graph of size k with $\gamma'_m(H_1) = \eta(k)$ and let H_2 be a graph of size $m - k$ such that $V(H_1) \cap V(H_2) = \emptyset$. Suppose $G = H_1 \cup H_2$ and f is a $\gamma'_m(H_1)$-function. Then $g : E(G') \longrightarrow \{-1, 0, 1\}$ defined by $g(e) = f(e)$ if $e \in E(H_1)$ and $g(e) = -1$ if $e \in E(H_2)$, is an MEkSDF of G with $g(E(G)) = \eta(k) - (m - k)$. This completes the proof. $\square$

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