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Changing of the Number of Minimum Dominating Sets after Edge Addition: Non Critical Edges

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Abstract

Let $\gamma(G)$ and $\#\gamma(G)$ denote the domination number and the number of all distinct minimum dominating sets of a graph G, respectively. We show that $\#\gamma(G+e) \geq \#\gamma(G)$ for every edge $e \in E(\overline{G})$ with $\gamma(G+e) = \gamma(G)$. Based on this, we begin an investigation of graphs for which $\#\gamma$ increases whenever an edge is added (we call such graphs $\#\gamma^+$-EA-critical). We prove that if G is $\#\gamma^+$-EA-critical then either G is edgeless or $\gamma(G+e) = \gamma(G)$ for all $e \in E(\overline{G})$. If p is a number of endvertices of a $\#\gamma^+$-EA-critical graph G of order $n \geq 6$, we show that (a) $p \leq \gamma(G)$, and (b) $p \leq \lfloor n/3 \rfloor$ provided G is connected. In both cases, we find the extremal graphs. We also find all $\#\gamma^+$-EA-critical graphs each component of which is either a tree or a unicyclic graph.

Keywords: Domination number, critical edge.

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [6]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For any vertex v of G, the *open neighborhood* of v is the set $N(v,G) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighborhood* of v is the set $N[v,G] = N(v,G) \cup \{v\}$. The *degree* of v is defined as $deg(v,G) = |N(v,G)|$. The *maximum* and *minimum degree* of vertices in $V(G)$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $\Delta(G) = \delta(G) = k$ then G is said to be *k-regular*. For a set of vertices $S \subseteq V(G)$, $N(S,G)$ is the union of $N(x,G)$ for all $x \in S$, and $N[S,G] = N(S,G) \cup S$. A cycle on n vertices is denoted by C_n and a path on m vertices by P_m. A *key* $L_{n,m}$ is the graph of order and size $m+n$ obtained by joining one vertex of a cycle C_n to one of the end-vertices of a path P_m. The girth of a graph is the length of its shortest cycle. If a graph has no cycles then its girth is said to be infinite. For disjoint graphs G and H, the *join* $G+H$ is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

A *dominating set* in a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G is either in D or is adjacent to an element of D. A dominating set D of a graph G is a *minimal dominating set* if no set $D' \subsetneq D$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. The set of all minimal dominating sets each of which has $U \subseteq V(G)$ as a subset is denoted by $MDS(U,G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Any dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-*set*, or just γ-set when the graph G is clear from the context. The set of all γ-sets of a graph G is denoted by $\mathscr{D}(G)$. If $U \subseteq V(G)$, then denote $\mathscr{D}(U,G) = \{M \in \mathscr{D}(G) : U \subseteq M\}$. The number of distinct $\gamma(G)$-sets is denoted $\#\gamma(G)$ ([15]). The number of all $\gamma(G)$-sets each of which has $U \subseteq V(G)$ as a subset is denoted by $\#\gamma(U,G)$. If $e \in \overline{G}$ then $\gamma(G) - 1 \leq \gamma(G+e) \leq \gamma(G)$ ([6]); if $\gamma(G+e) = \gamma(G) - 1$ then e is called a $\gamma(G)$-*critical edge*.

It is often of interest to known how the value of a graph parameter is affected when a small change is made in a graph, for instance vertex or edge removal, edge addition and edge contraction. The study of effects on domination related parameters when a graph is modified by adding an edge is classical; see for instance [5, 11, 14, 17] and for surveys [6, Chapter 5] and [3, 16]. In this connection, the present author [12] initiates an investigation of the changing and unchanging of $\#\gamma(G)$ in the case when any $\gamma(G)$-critical edge e is added to G. One remaining question to consider is what happens in the case when $e \in E(\overline{G})$ is non $\gamma(G)$-critical. In Section 2 we prove that $\#\gamma(G) \leq \#\gamma(G+e)$ whenever $\gamma(G+e) = \gamma(G)$ and determine sufficient conditions for validity of $\#\gamma(G) = \#\gamma(G+e)$ (resp. $\#\gamma(G) < \#\gamma(G+e)$). Motivated by these results we define:

Definition 1.1. *Let G be a graph. An edge $e \in E(\overline{G})$ is $\#\gamma^+(G)$-EA-critical if $\#\gamma(G) < \#\gamma(G+e)$. A graph G is $\#\gamma^+$-EA-critical if all edges of $\overline{G}$ are $\#\gamma^+(G)$-EA-critical.*

In Section 3 we begin an investigation of $\#\gamma^+$-EA-critical graphs.

In what follows we use the notation proposed in [13]. We need the following definitions and results. A vertex $v \in V(G)$ is called:

(a) [4] γ-*good*, if v belongs to some $\gamma(G)$-set;

(b) [4] γ-*bad*, if v belongs to no $\gamma(G)$-set;

(c) [14] γ-*fixed* if v belongs to every $\gamma(G)$-set;

(d) [14] γ-*free* if v belongs to some $\gamma(G)$-set but not to all $\gamma(G)$-sets.

$\mathbf{G}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-good }\}$;

$\mathbf{B}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-bad }\}$;

$\mathbf{Fi}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-fixed }\}$;

$\mathbf{Fr}(G) = \{x \in V(G) : x \text{ is } \gamma\text{-free }\}$;

$\mathbf{V}^0(G) = \{x \in V(G) : \gamma(G - x) = \gamma(G)\}$;

$\mathbf{V}^-(G) = \{x \in V(G) : \gamma(G - x) = \gamma(G) - 1\}$;

$\mathbf{V}^+(G) = \{x \in V(G) : \gamma(G - x) > \gamma(G)\}$.

Clearly $\{\mathbf{G}(G), \mathbf{B}(G)\}$ and $\{\mathbf{V}^-(G), \mathbf{V}^0(G), \mathbf{V}^+(G)\}$ are partitions of $V(G)$, and $\{\mathbf{Fi}(G), \mathbf{Fr}(G)\}$ is a partition of $\mathbf{G}(G)$ ([6]).

$\mathbf{Fr}^-(G) = \{x \in \mathbf{Fr}(G) : \gamma(G - x) = \gamma(G) - 1\}$;

$\mathbf{Fr}^0(G) = \{x \in \mathbf{Fr}(G) : \gamma(G - x) = \gamma(G)\}$;

$\mathbf{Fi}^p(G) = \{x \in \mathbf{Fi}(G) : \gamma(G - x) = \gamma(G) + p\}$, $p \in \{-1, 0, 1, .., |V(G)| - 2\}$.

Observation 1.2. [11] *If G is a graph of order $n \geq 2$ then:*

(1) $\{\mathbf{Fr}^-(G), \mathbf{Fr}^0(G)\}$ *is a partition of* $\mathbf{Fr}(G)$*;*
(2) $\{\mathbf{Fi}^{-1}(G), \mathbf{Fi}^0(G), \dots, \mathbf{Fi}^{n-2}(G)\}$ *is a partition of* $\mathbf{Fi}(G)$*;*
(3) $\{\mathbf{Fi}^{-1}(G), \mathbf{Fr}^-(G)\}$ *is a partition of* $\mathbf{V}^-(G)$*;*
(4) $\{\mathbf{Fi}^0(G), \mathbf{Fr}^0(G), \mathbf{B}(G)\}$ *is a partition of* $\mathbf{V}^0(G)$*;*
(5) $\{\mathbf{Fi}^1(G), \mathbf{Fi}^2(G), \dots, \mathbf{Fi}^{n-2}(G)\}$ *is a partition of* $\mathbf{V}^+(G)$*.*

Lemma 1.3. [2] *Let G be a graph.*

(i) If $v \in \mathbf{V}^+(G)$ then for every $\gamma(G)$-set S, the set $N(v, G) - N[S - \{v\}, G]$ contains at least two nonadjacent vertices.

(ii) $|\mathbf{V}^0(G)| \geq 2|\mathbf{V}^+(G)|$.

Lemma 1.4. *Let G be a graph.*

(i) ([11], Lemma 6(2)) If $x_1 \in \mathbf{V}^0(G)$ and $x_2 \in \mathbf{Fi}^q(G)$, $q \geq 2$, then $x_2 \in \mathbf{Fi}(G - x_1) - \mathbf{Fi}^{-1}(G - x_1)$.

(ii) ([11], Fact 1 in the proof of Theorem 4.1) Let x_1 and x_2 be distinct and nonadjacent vertices of G. Then $\gamma(G) = \gamma(G + x_1x_2)$ if and only if one of the following conditions holds:

(A_0) neither x_1 nor x_2 is in $\mathbf{V}^-(G)$;

(A_1) $x_1, x_2 \in \mathbf{V}^-(G)$, $x_1 \in \mathbf{B}(G - x_2)$ and $x_2 \in \mathbf{B}(G - x_1)$;

(A_2) $x_i \in \mathbf{V}^-(G)$, $x_j \notin \mathbf{V}^-(G)$ and $x_j \in \mathbf{B}(G - x_i)$, where $\{i, j\} = \{1, 2\}$.

(iii) ([11], Lemma 6(1)) If $x_1 \in \mathbf{Fr}^-(G)$ then $\mathbf{B}(G) \cup N(x_1, G) \subseteq \mathbf{B}(G - x_1)$.

(iv) ([11], Corollary 10 (a1)) If $x_1 \in \mathbf{Fi}^0(G)$ then $N(x_1, G) \subseteq \mathbf{B}(G - x_1) \cap (\mathbf{V}^0(G) \cup \mathbf{Fi}^1(G))$.

The following refinement of the condition (A_0) is necessary for the sequel.

Observation 1.5. *Let G be a graph, $x_1, x_2 \in V(G)$, $x_1 \neq x_2$ and $x_1x_2 \notin E(G)$. Then the conditions (B_1) – (B_7) together are equivalent to (A_0), where*

(B_1) $x_1 \in \mathbf{Fi}^p(G)$ *and* $x_2 \in \mathbf{Fi}^q(G)$, *where* $p, q \geq 1$;

(B_2) $x_i \in \mathbf{V}^0(G)$ *and* $x_j \in \mathbf{Fi}^q(G)$, $q \geq 2$, *where* $\{i, j\} = \{1, 2\}$;

(B_3) $x_i \in \mathbf{V}^0(G)$ *and* $x_j \in \mathbf{Fi}^1(G) \cap \mathbf{G}(G - x_i)$, *where* $\{i, j\} = \{1, 2\}$;

(B_4) $x_i \in \mathbf{V}^0(G)$ *and* $x_j \in \mathbf{Fi}^1(G) \cap \mathbf{B}(G - x_i)$, *where* $\{i, j\} = \{1, 2\}$;

(B_5) $x_1, x_2 \in \mathbf{V}^0(G)$, $x_1 \in \mathbf{B}(G - x_2)$ *and* $x_2 \in \mathbf{B}(G - x_1)$;

(B_6) $x_1, x_2 \in \mathbf{V}^0(G)$, $x_i \in \mathbf{B}(G - x_j)$ *and* $x_j \in \mathbf{G}(G - x_i)$, *where* $\{i, j\} = \{1, 2\}$;

(B_7) $x_1, x_2 \in \mathbf{V}^0(G)$, $x_1 \in \mathbf{G}(G - x_2)$ *and* $x_2 \in \mathbf{G}(G - x_1)$.

Lemma 1.6. [12] *Let G be a graph, $x_1, x_2 \in V(G)$ and $x_1x_2 \in E(\overline{G})$. If x_1x_2 is $\gamma(G)$-critical then one of the following holds:*

(i) at most one of x_1 and x_2 is isolated and $\#\gamma(G + x_1x_2) \leq \#\gamma(G)$;

(ii) both x_1 and x_2 are isolated and $\#\gamma(G + x_1x_2) = 2\#\gamma(G)$.

Moreover, if x_1 and x_2 are both nonisolated then x_1x_2 is $\gamma(G)$-critical if and only if $\#\gamma(G+ x_1x_2) < \#\gamma(G)$.

2. Minimum Dominating Sets

In this section we investigate relationships between $\mathscr{D}(G)$ and $\mathscr{D}(G+e)$, where $e \in E(\overline{G})$ is non $\gamma(G)$-critical.

Let G be a graph, $u, v \in V(G)$, $u \neq v$ and $uv \notin E(G)$. Define: $S(u, v) = \{M : M$ is a dominating set of $G - u, v \in M, |M| = \gamma(G)$ and $M \notin \mathscr{D}(G)\}$.

Theorem 2.1. *Let G be a graph, $x_1, x_2 \in V(G)$, $x_1x_2 \notin E(G)$ and $\gamma(G + x_1x_2) = \gamma(G)$. Then*

(i) $\mathscr{D}(G), S(x_1, x_2)$ and $S(x_2, x_1)$ are pairwise disjoint and $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup S(x_1, x_2) \cup S(x_2, x_1)$.

(ii) $\#\gamma(G + x_1x_2) \geq \#\gamma(G)$.

Proof. (i) By the definitions of $S(x_1, x_2)$ and $S(x_2, x_1)$ it immediately follows that: (a) $\mathscr{D}(G), S(x_1, x_2)$ and $S(x_2, x_1)$ are mutually disjoint sets, and (b) each element of $S(x_1, x_2) \cup S(x_2, x_1)$ is a dominating set of $G + x_1x_2$. Hence $\gamma(G + x_1x_2) = \gamma(G)$ implies that $\mathscr{D}(G) \cup S(x_1, x_2) \cup S(x_2, x_1) \subseteq \mathscr{D}(G + x_1x_2)$.

Let now $M \in \mathscr{D}(G + x_1x_2) - \mathscr{D}(G)$. Then $|M| = \gamma(G + x_1x_2) = \gamma(G)$ and $|M \cap \{x_1, x_2\}| = 1$. This implies $M \in S(x_1, x_2) \cup S(x_2, x_1)$. Hence $\mathscr{D}(G + x_1x_2) - \mathscr{D}(G) \subseteq S(x_1, x_2) \cup S(x_2, x_1)$.

(ii) Immediately by (i). □

We need the following lemma to give some sufficient conditions for an edge of a graph $\overline{G}$ to be $\#\gamma^+(G)$-EA-critical (non $\#\gamma^+(G)$-EA-critical, respectively).

Lemma 2.2. *Let G be a graph, $x_1, x_2 \in V(G)$, $x_1x_2 \notin E(G)$ and $\gamma(G + x_1x_2) = \gamma(G)$.*

(i) If $x_1 \in \mathbf{Fi}^q(G)$, $q \geq 1$, then $S(x_1, x_2) = \emptyset$.

(ii) Let $x_1 \in \mathbf{V}^0(G)$. Then $S(x_1, x_2) = \mathscr{D}(\{x_2\}, G - x_1) - \mathscr{D}(G)$. Moreover,

(ii.1) if $x_2 \in \mathbf{B}(G - x_1)$ then $S(x_1, x_2) = \emptyset$;

(ii.2) if $x_2 \in \mathbf{Fi}^s(G)$, $s \geq 2$, then $S(x_1, x_2) = \mathscr{D}(G - x_1) - \mathscr{D}(G)$;

(ii.3) if $x_1 \in \mathbf{Fi}^0(G)$ then $S(x_1, x_2) = \mathscr{D}(\{x_2\}, G - x_1)$;

(ii.4) if $x_2 \in \mathbf{B}(G) \cap \mathbf{G}(G - x_1)$ then $S(x_1, x_2) = \mathscr{D}(\{x_2\}, G - x_1) \neq \emptyset$.

(iii) Let $x_1 \in \mathbf{V}^-(G)$, $A = \{Q \cup \{x_2\} : Q \in \mathscr{D}(G - x_1)\}$ and $B = \{M : M \in MDS(\{x_2\}, G - x_1), |M| = \gamma(G)$ and $M \cap N(x_1, G) = \emptyset\}$. Then $x_2 \in \mathbf{B}(G - x_1)$, $\#\gamma(G - x_1) = |A| \neq 0$, $A \cap B = \emptyset$ and $S(x_1, x_2) = A \cup B \neq \emptyset$.

Proof. (i) If M is a dominating set of $G - x_1$, then $|M| \geq \gamma(G - x_1) = \gamma(G) + q > \gamma(G)$.

(ii) Since $\gamma(G - x_1) = \gamma(G)$, $S(x_1, x_2) = \{M : M$ is a $\gamma(G - x_1)$-set, $x_2 \in M$ and $M \notin \mathscr{D}(G)\} = \mathscr{D}(\{x_2\}, G - x_1) - \mathscr{D}(G)$.

(ii.1) Clearly $x_2 \in \mathbf{B}(G - x_1)$ implies $\mathscr{D}(\{x_2\}, G - x_1) = \emptyset$; hence $S(x_1, x_2) = \emptyset$.

(ii.2) By Lemma 1.4(i), $x_2 \in \mathbf{Fi}(G - x_1)$; hence $\mathscr{D}(\{x_2\}, G - x_1) = \mathscr{D}(G - x_1)$.

(ii.3)–(ii.4) Obvious.

(iii) By Lemma 1.4(ii), $x_2 \in \mathbf{B}(G - x_1)$. By the definition of the set A, $|A| = |\mathscr{D}(G - x_1)| \neq 0$ and no member of A is in $MDS(G - x_1)$, which implies $A \cap B = \emptyset$. It remains to prove that $S(x_1, x_2) = A \cup B$. Let $M \in A$. Then $x_2 \in M$ and M is a dominating set of $G - x_1$ with $|M| = \gamma(G - x_1) + 1 = \gamma(G)$. If M is a dominating set of G, then $M - \{x_2\}$ is a dominating set of G too, with $|Q| < \gamma(G)$ - a contradiction. Hence $A \subseteq S(x_1, x_2)$ and since $A \neq \emptyset$ and clearly $B \subseteq S(x_1, x_2)$, it follows that $\emptyset \neq A \cup B \subseteq S(x_1, x_2)$.

Consider now any set $M \in S(x_1, x_2) - MDS(G - x_1)$. Then there exists a set $Q \in MDS(G - x_1)$ with $Q \subsetneq M$. By $\gamma(G) - 1 = \gamma(G - x_1) \leq |Q| < |M| = \gamma(G)$ it follows that $Q \in \mathscr{D}(G - x_1)$. Hence $M = Q \cup \{x_2\} \in A$. Now, let $M \in S(x_1, x_2) \cap MDS(G - x_1)$. Then $x_2 \in M$, $|M| = \gamma(G)$ and since $M \notin \mathscr{D}(G)$ it follows that $M \cap N(x_1, G) = \emptyset$. This implies $M \in B$. Thus $A \cup B \supseteq S(x_1, x_2)$. □

Theorem 2.3. *Let G be a graph, $x_1, x_2 \in V(G) - \mathbf{V}^-(G)$ and $x_1x_2 \in E(\overline{G})$. Then $\gamma(G + x_1x_2) = \gamma(G)$ and one of $(B_1) - (B_7)$ stated in Observation 1.5 holds.*

(i) If one of $(B_1), (B_4)$ and (B_5) holds then $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G)$ and $\#\gamma(G + x_1x_2) = \#\gamma(G)$.

(ii) If (B_2) holds then $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_j\}, G - x_i) = \mathscr{D}(G) \cup \mathscr{D}(G - x_i)$.

(iii) If one of (B_3) *and* (B_6) *holds then* $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_j\}, G - x_i)$.

(iv) If (B_7) *holds then* $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_2\}, G - x_1) \cup \mathscr{D}(\{x_1\}, G - x_2)$.

Proof. By Lemma 1.4(ii), $\gamma(G + x_1x_2) = \gamma(G)$ and by Observation 1.5, at least one of (B_1)–(B_7) holds.

Recall that $\mathscr{D}(G + x_1, x_2) = \mathscr{D}(G) \cup S(x_1, x_2) \cup S(x_2, x_1)$, because of Theorem 2.1.

(i) If (B_1) holds then by Lemma 2.2(i), $S(x_1, x_2) = S(x_2, x_1) = \emptyset$.

If (B_4) holds then $S(x_j, x_i) = \emptyset$ and $S(x_i, x_j) = \emptyset$ by Lemma 2.2(i) and (ii.1), respectively.

If (B_5) holds, by Lemma 2.2 (ii.1) we have $S(x_1, x_2) = S(x_2, x_1) = \emptyset$.

(ii) Let (B_2) hold. By Lemma 1.4 (i), $x_j \in \mathbf{Fi}(G - x_i)$ which implies $\mathscr{D}(\{x_j\}, G - x_i) = \mathscr{D}(G - x_i)$. Hence $S(x_i, x_j) = \mathscr{D}(G - x_i) - \mathscr{D}(G)$ because of Lemma 2.2 (ii). Finally, it follows by Lemma 2.2 (i) that $S(x_j, x_i) = \emptyset$.

(iii) If (B_3) holds then $S(x_i, x_j) = \mathscr{D}(\{x_j\}, G - x_i) - \mathscr{D}(G)$ (by Lemma 2.2(ii)) and $S(x_j, x_i) = \emptyset$ (by Lemma 2.2(i)).

If (B_6) holds then by Lemma 2.2(ii) we have $S(x_i, x_j) = \mathscr{D}(\{x_j\}, G - x_i) - \mathscr{D}(G)$ and $S(x_j, x_i) = \emptyset$.

(iv) $S(x_1, x_2) = \mathscr{D}(\{x_2\}, G - x_1) - \mathscr{D}(G)$ and $S(x_2, x_1) = \mathscr{D}(\{x_1\}, G - x_2) - \mathscr{D}(G)$ because of Lemma 2.2(ii). □

The following theorem gives sufficient conditions for an edge to be $\#\gamma^+$-EA-critical.

Theorem 2.4. *Let* G *be a graph,* $x_1, x_2 \in V(G)$, $x_1 \neq x_2$, $x_1x_2 \notin E(G)$ *and let one of the conditions* (A_1)–(A_5) *hold, where*

(A₁) $x_1, x_2 \in \mathbf{V}^-(G)$, $x_1 \in \mathbf{B}(G - x_2)$ *and* $x_2 \in \mathbf{B}(G - x_1)$*;*

(A₂) $x_i \in \mathbf{V}^-(G)$, $x_j \notin \mathbf{V}^-(G)$ *and* $x_j \in \mathbf{B}(G - x_i)$, *where* $\{i, j\} = \{1, 2\}$*;*

(A₃) $x_i \in \mathbf{Fi}^0(G)$, $x_j \in \mathbf{Fi}^q(G), q \geq 2$, *where* $\{i, j\} = \{1, 2\}$*;*

(A₄) $x_i \in \mathbf{Fi}^0(G)$, $x_j \in (\mathbf{Fi}^1(G) \cup \mathbf{V}^0(G)) \cap \mathbf{G}(G - x_i)$, *where* $\{i, j\} = \{1, 2\}$*;*

(A₅) $x_i \in \mathbf{B}(G) \cap \mathbf{G}(G - x_j)$, $x_j \in \mathbf{V}^0(G)$, *where* $\{i, j\} = \{1, 2\}$.

Then $\gamma(G + x_1x_2) = \gamma(G)$ *and* $\#\gamma(G + x_1x_2) > \#\gamma(G)$.

Proof. If one of $(A_3) - (A_5)$ holds then (A_0) stated in Lemma 1.4 (ii) is valid. Hence Lemma 1.4 (ii) leads to $\gamma(G + x_1x_2) = \gamma(G)$. To prove $\#\gamma(G + x_1x_2) > \#\gamma(G)$ it is sufficient to show that $S(x_1, x_2) \cup S(x_2, x_1) \neq \emptyset$, because of Theorem 2.1.

It follows by Lemma 2.2 (iii) that $S(x_1, x_2) \neq \emptyset$ when (A_1) holds, and $S(x_i, x_j) \neq \emptyset$ when (A_2) holds.

If (A_3) or (A_4) holds then by Lemma 2.2 (ii.3), $S(x_i, x_j) = \mathscr{D}(\{x_j\}, G - x_i)$. Since $x_j \in \mathbf{G}(G - x_i)$ (by Lemma 1.4 (i) when (A_3) holds), $\mathscr{D}(\{x_j\}, G - x_i) \neq \emptyset$.

Finally, if (A_5) holds then Lemma 2.2(ii.4) implies $S(x_j, x_i) \neq \emptyset$. □

3. $\#\gamma^+$-EA-critical graphs

Recall that a graph G is $\#\gamma^+$-EA-critical if $\#\gamma(G + e) > \#\gamma(G)$ for all $e \in E(\overline{G})$. We begin with easy observations.

Observation 3.1. *Let G be a graph with $n \geq 2$ components.*

(i) If G is $\#\gamma^+$-EA-critical then the union of any $k \leq n$ its components is a $\#\gamma^+$-EA-critical graph.

(ii) G is $\#\gamma^+$-EA-critical if and only if the union of each two components of G is $\#\gamma^+$-EA-critical.

Observation 3.2. *Let G be a graph with $\gamma(G) = 1$. Then G is $\#\gamma^+$-EA-critical if and only if either G is complete or each edge of $\overline{G}$ is incident with a vertex of degree $|V(G)| - 2$ in G.*

By (UEA) we denote the class of all graphs G with $\gamma(G) = \gamma(G+e)$ whenever $e \in E(\overline{G})$. Surveys and results on (UEA)-graphs may be found in [3, 5, 6, 11].

Theorem 3.3. *Let G be a graph. If $e \in E(\overline{G})$ is $\#\gamma^+$-EA-critical then either $\gamma(G + e) = \gamma(G)$ or the ends of e are isolated in G. If G is $\#\gamma^+$-EA-critical then either G is edgeless or G is in (UEA).*

Proof. The results immediately follow by Lemma 1.6. □

Observe that not all (UEA)-graphs are $\#\gamma^+$-EA-critical. For example all stars $K_{1,s}$ where $s \geq 3$ are in (UEA) but none of them is $\#\gamma^+$-EA-critical.

A graph is $2K_2$-free if and only if it does not contain $2K_2$ as an induced subgraph.

Theorem 3.4. *Let G be a $(n-3)$-regular graph of order $n \geq 3$. Then G is $\#\gamma^+$-EA-critical if and only if G is $2K_2$-free.*

Proof. If $n \in \{3, 4\}$ then $G \in \{\overline{K_3}, 2K_2\}$. So, let $n \geq 5$. It is well known, that if a graph H has no isolated vertices, then $\gamma(H) \leq (|V(H)| + 2 - \delta(H))/2$ (Payan [9]). Hence $\gamma(G) = 2$ and $V(G) = \mathbf{V}^0(G)$.

First assume that there is a set $\{x_1, x_2, x_3, x_4\} \subset V(G)$ that induces $2K_2$ and $x_1x_3, x_2x_4 \in E(G)$. Then $\{x_1, x_4\} \cup \cup_{i=1}^{n-4}\{x_1, z_i\} = \mathscr{D}(\{x_1\}, G - x_2) \subseteq \mathscr{D}(G)$ where

$\{z_1,..,z_{n-4}\} = V(G) - \{x_1,..,x_4\}$. Similarly we obtain $\mathscr{D}(\{x_2\}, G - x_1) \subseteq \mathscr{D}(G)$ Since (B_7) holds, by Theorem 2.3(iv) we have $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G)$.

Now, assume that G is $2K_2$-free. Let $x \in V(G)$ be arbitrary and let x_1 and x_2 be the two vertices outside $N[x, G]$.

Case 1. $x_1x_2 \in E(G)$. Let $y_1 \in V(G)$ be such that $y_1x \in E(G)$ and $y_1x_1 \notin E(G)$. Since G is $2K_2$-free, y_1 and x_2 are adjacent. Hence $\{x, y_1\}$ is not a $\gamma(G)$-set but it is a $\gamma(G + xx_1)$-set.

Case 2. $x_1x_2 \notin E(G)$. Hence $N(x_1, G) = N(x_2, G) = N(x, G) = V(G) - \{x_1, x_2, x\}$. In this case, clearly, $\{x, x_2\}$ is not a $\gamma(G)$-set but it is a $\gamma(G + xx_1)$-set. □

Remark 3.5. *Let G be a $(n-3)$-regular graph of order $n \geq 7$. Then G has an induced $2K_2$ if and only if $G = 2K_2 + H$ where H is a $(n-7)$-regular graph of order $n-4$.*

Proposition 3.6. *Let G be a $\#\gamma^+$-EA-critical graph with at least one edge. Then:*

(1) $\mathbf{V}^-(G) = \emptyset$;

(2) $\mathbf{Fi}(G) = \mathbf{Fi}^0(G) \cup \mathbf{Fi}^1(G)$;

(3) $\langle \mathbf{Fi}^1(G), G) \rangle$ *is complete provided* $\mathbf{Fi}^1(G) \neq \emptyset$;

(4) if $u \in \mathbf{Fi}^1(G)$ then u has at most 2 endvertices as neighbors;

(5) no endvertex is adjacent to a vertex in $\mathbf{Fi}^0(G)$.

Proof. (1) Assume to the contrary that $x \in \mathbf{V}^-(G)$ and let M be a $\gamma(G-x)$-set. Now, for any $y \in M$, M is a dominating set of $G + xy$. Hence $\gamma(G + xy) \leq |M| = \gamma(G - x) < \gamma(G)$ which is a contradiction because of Theorem 3.3.

(2) Let $x \in \mathbf{V}^+(G)$ and $M \in \mathscr{D}(G)$. Then there exist two nonadjacent vertices $x_1, x_2 \in N(x, G) - M$ (by Lemma 1.3). It follows by (1) that $x_1, x_2 \in \mathbf{V}^0(G)$ and then one of $(B_5), (B_6)$ and (B_7) holds. Theorem 2.3 implies that $\gamma(G) = \gamma(G + x_1x_2)$ and there is $R \in (\mathscr{D}(G + x_1x_2) - \mathscr{D}(G)) \cap \mathscr{D}(\{x_i\}, G - x_j)$ where $\{i, j\} = \{1, 2\}$. Note that $x \notin R$ - otherwise $R \in \mathscr{D}(G)$. Now by Lemma 1.4 (i) we yields $\mathbf{Fi}^q(G) = \emptyset$ whenever $q \geq 2$. It remains to note that by (1), $\mathbf{Fi}^{-1}(G) = \emptyset$.

(3) Assume that $x_1, x_2 \in \mathbf{Fi}^1(G)$ are nonadjacent. Then (B_1) holds and Theorem 2.3(i) implies $\#\gamma(G) = \#\gamma(G + x_1x_2)$ - a contradiction.

(4) Assume to the contrary that $x_1, x_2, x_3 \in N(u, G)$ are mutually disjoint endvertices. Then, clearly, (B_5) holds and Theorem 2.3(i) leads to a contradiction.

(5) It immediately follows by Lemma 1.4(iv). □

Lemma 3.7. *Let G be a graph, $x \in \mathbf{Fr}^0(G)$ and $\{y\} = N(x, G)$. Then $y \in \mathbf{Fr}^0(G) \cap \mathbf{V}^-(G - x)$.*

Proof. Let M be any $\gamma(G)$-set with $x \in M$. Then $y \notin M$ and $M_1 = (M - \{x\}) \cup \{y\}$ is a $\gamma(G)$-set. Hence $y \in \mathbf{Fr}(G)$. If Q is a $\gamma(G-y)$-set then $x \in Q$ and Q is a dominating set of G; hence $\gamma(G) \leq |Q| = \gamma(G-y)$. Thus $y \in \mathbf{Fr}^0(G)$. Moreover, $M - \{x\}$ is a dominating set of $G - \{x,y\}$. So we have $\gamma((G-x)-y) = \gamma(G - \{x,y\}) \leq |M - \{x\}| = \gamma(G) - 1 = \gamma(G-x) - 1$ which leads to $y \in \mathbf{V}^-(G-x)$. □

In the next theorems we concentrate on $\mathbf{P}(G)$ - the set of all endvertices of a $\#\gamma^+$-EA-critical graph G.

Theorem 3.8. *Let G be a $\#\gamma^+$-EA-critical graph.*

(i) Then $\mathbf{P}(G) \subseteq \mathbf{B}(G) \cup \mathbf{Fr}^0(G)$.

(ii) Let $y_1, y_2 \in \mathbf{P}(G)$ and $y_1 \in \mathbf{Fr}^0(G)$. Then $y_1y_2 \in E(G)$, $\mathbf{P}(G) = \{y_1, y_2\} \subseteq \mathbf{Fr}^0(G)$ and $\mathbf{Fi}^1(G) = \emptyset$.

(iii) If $x_1, y_1 \in \mathbf{P}(G)$ have a common neighbor then $\{x_1, y_1\} = \mathbf{P}(G) \subseteq \mathbf{B}(G)$.

(iv) At most one component of G has endvertices.

Proof. Let G have endvertices.

(i) Clearly, no endvertex is in $\mathbf{Fi}(G)$. The result follows since $\mathbf{Fr}^-(G)$ is empty by Proposition 3.6(1).

(ii) Assume to the contrary that $y_1y_2 \notin E(G)$. Denote x_i to be the unique neighbor of y_i, $i = 1, 2$. Since $y_1 \in \mathbf{Fr}^0(G)$ we have that $x_1 \neq x_2$ and there is a $\gamma(G)$-set, say M, with $y_1 \in M$. Hence the set $M_1 = (M - \{y_2\}) \cup \{x_2\}$ is a $\gamma(G)$-set too (it is possible $M = M_1$). Note that x_1 and x_2 are nonadjacent in G - otherwise $M_1 - \{y_1\}$ is a dominating set of $G - y_1$, of cardinality $\gamma(G) - 1$, which implies $y_1 \in \mathbf{Fr}^-(G)$ - a contradiction because of Proposition 3.6(1). By Lemma 3.7, $x_1 \in \mathbf{Fr}^0(G)$. Since x_1 is adjacent to an endvertex in $G - x_2$, it follows that $x_1 \in \mathbf{G}(G - x_2)$. Analogously we have $x_2 \in \mathbf{G}(G - x_1)$. Note that by (i), $y_2 \in \mathbf{B}(G) \cup \mathbf{Fr}^0(G)$.

First assume $y_2 \in \mathbf{Fr}^0(G)$. Then each member of $\mathscr{D}(\{x_1\}, G - x_2)$ is a dominating set of G and since $x_2 \in \mathbf{Fr}^0(G)$ (it follows by Lemma 3.7), $\mathscr{D}(\{x_1\}, G - x_2) \subseteq \mathscr{D}(G)$. Similarly we obtain $\mathscr{D}(\{x_2\}, G - x_1) \subseteq \mathscr{D}(G)$. Thus (B_7) holds and by Theorem 2.3 (iv), it follows that $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G)$ - a contradiction.

Second, let $y_2 \in \mathbf{B}(G)$. Now Proposition 3.6(2),(5) leads to $x_2 \in \mathbf{Fi}^1(G)$. Hence (B_3) holds and Theorem 2.3 (iii) implies that $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_2\}, G - x_1)$. For any $M \in \mathscr{D}(\{x_2\}, G - x_1)$, $y_1 \in M$ which implies that M is a dominating set of G with $|M| = \gamma(G - x_1 = \gamma(G)$. Hence M is a $\gamma(G)$-set and $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G)$ - a contradiction.

Thus $y_1y_2 \in E(G)$ which leads to $\{y_1, y_2\} = \mathbf{P}(G) \cap \mathbf{Fr}^0(G)$. Note that $\mathscr{D}(G) = \mathscr{D}(G - y_1) \cup \mathscr{D}(G - y_2)$.

Assume to the contrary that there is $z \in \mathbf{Fi}^1(G)$. Then $\mathscr{D}(G) = \mathscr{D}(\{z\}, G - y_1) \cup \mathscr{D}(\{z\}, G - y_2)$. Since (B_3), applied to z and y_1, holds it follows by Theorem 2.3 (iii) that $\mathscr{D}(G + y_1 z) = \mathscr{D}(G) \cup \mathscr{D}(\{z\}, G - y_1) = \mathscr{D}(G)$ - a contradiction. Thus $\mathbf{Fi}^1(G)$ is empty.

Finaly, if $u \in \mathbf{P}(G) \cap \mathbf{B}(G)$ then its unique neighbor is in $\mathbf{Fi}^1(G)$ (because of Proposition 3.6 (2), (5)) - a contradiction.

(iii) Let $\{z\} = N(x_1, G)$. Obviously $z \in \mathbf{Fi}(G)$ and $\{x_1, y_1\} \subseteq \mathbf{B}(G)$. Moreover, $z \in \mathbf{Fi}^1(G)$ because of Proposition 3.6(2),(5). Assume to the contrary that there is $x_2 \in \mathbf{P}(G) - \{x_1, y_1\}$. By Proposition 3.6(4), $x_2 z \notin E(G)$ and let $N(x_2, G) = \{w\}$. By (i) and (ii), $x_2 \in \mathbf{B}(G)$. By Proposition 3.6 (2),(5) it follows that $w \in \mathbf{Fi}^1(G)$ and then Proposition 3.6 (3) leads to $wz \in E(G)$. Clearly, $x_1 \in \mathbf{B}(G - x_2)$. Hence one of (B_5) and (B_6) holds. Theorem 2.3 (i) implies that (B_6) holds and now, by Theorem 2.3(iii), $\mathscr{D}(G + x_1 x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_2\}, G - x_1)$. Consider any set $M \in \mathscr{D}(\{x_2\}, G - x_1)$. Since $x_2 \in M$ is an endvertex, $w \notin M$. Let $M_1 = (M - \{y_1\}) \cup \{z\}$ (it is possible that $M_1 = M$). Obviously M_1 is a $\gamma(G)$-set and since $w \notin M_1$, this contradicts $w \in \mathbf{Fi}(G)$. Thus $\mathscr{D}(\{x_2\}, G - x_1) = \emptyset$ and the result follows.

(iv) Assume to the contrary, that $x_1, x_2 \in \mathbf{P}(G)$ belong to distinct components of G. By (i) and (ii) it follows that $x_1, x_2 \in \mathbf{B}(G)$ and therefore $y_1, y_2 \in \mathbf{Fi}(G)$ where $\{y_i\} = N(x_i, G)$, $i = 1, 2$. However, by Proposition 3.6(2),(5), we have $y_1, y_2 \in \mathbf{Fi}^1(G)$ and by Proposition 3.6(3), $y_1 y_2 \in E(G)$ - a contradiction. □

$\#\gamma^+$-EA-critical graphs of girth at least 4 have at most two endvertices, as it is shown in the next corollary.

Corollary 3.9. *Let G be a $\#\gamma^+$-EA-critical graph of girth at least* 4. *Then* $|\mathbf{P}(G)| \leq 2$. *If G is a forest then either G is edgeless or $G \in \{K_2, K_{1,2}\}$.*

Proof. Let $y_1 \in \mathbf{P}(G)$. Note that $\mathbf{V}^-(G)$ is empty by Proposition 3.6(1). If $y_1 \in \mathbf{Fr}^0(G)$ then by Theorem 3.8(ii) it immediately follows that $|\mathbf{P}(G)| \leq 2$ and in case when the equality holds, $\langle \mathbf{P}(G), G \rangle$ is a component of G. Hence if G is a forest then $G = K_2$.

So, it remains the case $\mathbf{P}(G) \subseteq \mathbf{B}(G)$ because of Theorem 3.8(i). If there are two endvertices having a common neighbor in G then Theorem 3.8(iii) implies $|\mathbf{P}(G)| = 2$ and clearly if G is a forest then $G = K_{1,2}$.

Finally, let each two distinct endvertices of G be at distance at least three. Clearly $N(\mathbf{P}(G), G) \subseteq \mathbf{Fi}(G)$ and by Proposition 3.6 (5), (2) we have $N(\mathbf{P}(G), G) \subseteq \mathbf{Fi}^1(G)$. Since $\langle \mathbf{Fi}^1(G), G \rangle$ is complete (by Proposition 3.6(3)) and since G has girth at least 4, it follows that $|\mathbf{Fi}^1(G)| \leq 2$. This leads to $|\mathbf{P}(G)| \leq 2$ and if $\mathbf{P}(G) = \{x_1, x_2\}$ then x_1 and x_2 are at distance three in G. Now if G is a forest then $G = P_4$ which is clearly impossible. □

In order to formulate the next results we need the following definition.

Let $p \geq 2$ and $r \geq 0$ be integers. Let $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_p\}$, $Z = \{z_1, z_2, \ldots, z_p\}$ and if $r \geq 1$ then $T = \{t_1, t_2, \ldots, t_r\}$ (X, Y, Z and T are paired disjoint). Define the graphs $S_{p,r}$ as follows (see Figure 1):

• $V(S_{p,0}) = X \cup Y \cup Z$ and $E(S_{p,0}) = (\cup_{i=1}^{p}\{y_i x_i, y_i z_i\}) \cup E(\langle Y, S_{p,0}\rangle) \cup E(\langle Z, S_{p,0}\rangle)$, where $\langle Y, S_{p,0}\rangle$ and $\langle Z, S_{p,0}\rangle$ are complete graphs.

• $V(S_{p,r}) = V(S_{p,0}) \cup T$ and $E(S_{p,r}) = E(S_{p,0}) \cup (\cup_{i=1}^{p} \cup_{j=1}^{r} \{y_i t_j, t_j z_i\}) \cup E(\langle T, S_{p,r}\rangle)$ where $r \geq 1$ and $\langle T, S_{p,r}\rangle$ is a complete graph.

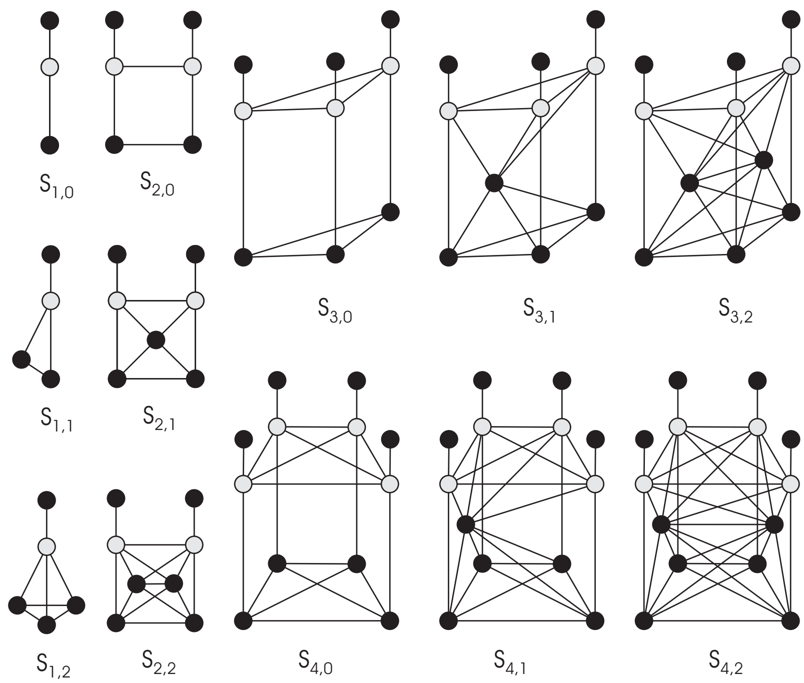

Figure 1: Graphs $S_{p,r}$ for small p and r. The light vertices form $\mathbf{Fi}^1(S_{p,r})$.

Observation 3.10. *$S_{p,r}$ is a $\#\gamma^+$-EA-critical graph. $\mathbf{Fi}^1(S_{p,r}) = \{y_1, \ldots, y_p\}$ is the unique $\gamma(S_{p,r})$-set.*

In general, $|\mathbf{P}(G)|$ and $\gamma(G)$ are incomparable. However, if G is $\#\gamma^+$-EA-critical, the following is valid.

Theorem 3.11. *Let G be a $\#\gamma^+$-EA-critical graph of order n and $\gamma(G) = \gamma$. Then $\gamma \geq |\mathbf{P}(G)|$ except for $G \in \{K_2, K_{1,2} = S_{1,0}\}$. Moreover, $\gamma = |\mathbf{P}(G)|$ if and only if $n \geq 4$ and $G = S_{\gamma, n-3\gamma}$.*

Proof. Let G have endvertices.

Case 1. $\gamma(G) = 1$. It follows by Observation 3.2 that either $n = 2$ and $G = K_2$ or $n > 2$ and $G = S_{1,n-3}$.

Case 2. $\gamma(G) \geq 2$ and there are at least two endvertices, say $x_1, \ldots, x_k$ with a common neighbor, say y. By Theorem 3.8(iii), $\mathbf{P}(G) = \{x_1, x_2\} \subseteq \mathbf{B}(G)$. So, let $\gamma(G) = 2$. Since $\mathbf{V}^-(G) = \emptyset$ and $\mathbf{Fi}(G) = \mathbf{Fi}^0(G) \cup \mathbf{Fi}^1(G)$ (Proposition 3.6(1),(2)), for each $u \in V(G) - \{y, x_1, x_2\}$ one of (B_3)–(B_7), applied to x_1 and u, holds. Note that $\{x_1, y\}$ does not dominate $G + ux_1$ - otherwise $u \in \mathbf{V}^-(G)$, a contradiction with Proposition 3.6(1). Now, Theorem 2.3 implies that one of (B_3) and (B_6) holds (applied to x_1 and u), and $\mathscr{D}(G + x_1u) = \mathscr{D}(G) \cup \mathscr{D}(\{u\}, G - x_1)$. Since G is $\#\gamma^+$-EA-critical, $\{u, x_2\}$ is in $\mathscr{D}(G + x_1u) - \mathscr{D}(G)$. Thus $G - \{y, x_1, x_2\}$ is complete which implies $\mathscr{D}(G) = \mathscr{D}(G + yw)$ for each $w \in V(G) - N[y, G]$ - a contradiction.

Case 3. $\gamma(G) \geq 2$ and no vertex in G is adjacent to at least two endvertices.

Let $\mathbf{P}(G) = \{x_1, \ldots, x_p\}$ and $\{y_i\} = N(x_i, G)$, $i = 1, \ldots, p$. Because of Theorem 3.8(i),(ii), either $p = 2$ and $x_1x_2 \in E(G)$ or $\mathbf{P}(G) \subseteq \mathbf{B}(G)$. In the last case, $\{y_1, \ldots, y_p\} \subseteq \mathbf{Fi}(G)$ and then $p \leq |\mathbf{Fi}(G)| \leq \gamma(G)$.

Now, let $\gamma(G) = |\mathbf{P}(G)|$. First assume $\gamma(G) = p = 2$ and $x_1x_2 \in E(G)$. Then G has exactly two components, namely $G_1 = \langle \mathbf{P}(G), G \rangle = K_2$ and G_2 with $\gamma(G_2) = 1$ and $|V(G)| \geq 3$ (Proposition 3.6(1)). But then for any vertex of maximum degree in G_2, say w, $\mathscr{D}(G) = \mathscr{D}(G + wx_1)$ - a contradiction.

It remains to consider the case $\gamma(G) = |\mathbf{P}(G)|$ and $\mathbf{P}(G) \subseteq \mathbf{B}(G)$. As we have already known, $\gamma(G) \geq |\mathbf{Fi}(G)| \geq p = |\mathbf{P}(G)|$ which implies $\{y_1, \ldots, y_p\}$ is the unique $\gamma(G)$-set and $V(G) - \{y_1, \ldots, y_p\} = \mathbf{B}(G)$. Moreover, by Proposition 3.6(2),(5) it follows that $\{y_1, \ldots, y_p\} = \mathbf{Fi}^1(G)$ and then by Proposition 3.6(3), $\mathbf{Fi}^1(G)$ induces a complete graph.

Let $u \in V_1 = V(G) - \{x_1, y_1, \ldots, x_p, y_p\}$. Since $u \in \mathbf{B}(G)$ and $\gamma(G - u) = \gamma(G)$ and since $x_1, \ldots, x_p$ are endvertices of $G - u$, for any $\gamma(G - u)$-set, say M, $|M \cap \{x_i, y_i\}| = 1$, $i = 1, \ldots, p$. Thus $V_1 - \{u\} \subseteq \mathbf{B}(G - u)$. Now, if there are nonadjacent vertices $u_1, u_2 \in V_1$ then (B_5), applied for u_1 and u_2, holds and Theorem 2.3(i) leads to a contradiction. Thus V_1 induces a complete graph.

It follows by Lemma 1.3(1) that there are $z_1, z_2, \ldots, z_p \in V(G) - \{x_1, \ldots, x_p\}$ such that $z_i \in N(y_i, G) - N[\mathbf{Fi}^1(G) - \{y_i\}, G]$, $i = 1, \ldots, p$. If there are nonadjacent y_k and $u \in V_1 - \{z_1, \ldots, z_p\}$ then one of (B_3) and (B_4), applied to u and y_k, holds and now Theorem 2.3(i),(iii) implies $\mathscr{D}(G + uy_k) = \mathscr{D}(G) \cup \mathscr{D}(\{y_k\}, G - u)$. But clearly $\{y_1, \ldots, y_p\}$ is the unique minimum dominating set of both G and $G - u$ - a contradiction. Hence for any $w \in V_1 - \{z_1, \ldots, z_p\}$, $N(w, G) \supset \{y_1, \ldots, y_p\}$. Thus $G = S_{\gamma, n-3\gamma}$. □

What is the maximum number of endvertices which a n-vertex $\#\gamma^+$-EA-critical graph can have? The following theorem gives the answer.

Theorem 3.12. *Let G be a connected $\#\gamma^+$-EA-critical graph of order $n \geq 6$. Then $|\mathbf{P}(G)| \leq \lfloor n/3 \rfloor$. Moreover $|\mathbf{P}(G)| = \lfloor n/3 \rfloor$ if and only if $G = S_{\lfloor n/3 \rfloor, n-3\lfloor n/3 \rfloor}$.*

Proof. Let $\mathbf{P}(G) = \{x_1, \ldots, x_p\}$.

Case 1. No vertex of G is adjacent to at least two endvertices.

Let $\{y_i\} = N(x_i, G)$, $i = 1, \dots, p$. Because of Theorem 3.8(i),(ii), we have $\mathbf{P}(G) \subseteq \mathbf{B}(G)$ and then $N(\mathbf{P}(G), G) \subseteq \mathbf{Fi}(G)$. Now, by Proposition 3.6(2),(5) it follows that $\{y_1, \dots, y_p\} \subseteq \mathbf{Fi}^1(G)$. Choose M to be an arbitrary $\gamma(G)$-set. Then by Lemma 1.3(i), there are $z_1, \dots, z_p$ such that $z_i \neq x_i$, $z_i x_i \notin E(G)$ and $z_i \in N(y_i, G) - N[M - \{y_i\}, G]$, $i = 1, \dots, p$. Hence $p = |\mathbf{P}(G)| \leq |\mathbf{Fi}^1(G)| = |\mathbf{V}^+(G)| \leq (|\mathbf{V}^+(G)| + |\mathbf{V}^0(G)|)/3 = n/3$ because of Proposition 3.6(1),(2) and Lemma1.3(ii). Moreover, $|\mathbf{P}(G)| = \lfloor n/3 \rfloor$ if and only if $\mathbf{Fi}^1(G) = \{y_1, \dots, y_p\}$.

Now, let $n = 3p + r$, $r \in \{0, 1, 2\}$. Assume that $\gamma(G) > p$. Since $\mathbf{V}^-(G)$ is not empty (Proposition 3.6(1)), $r = 2$ and the two vertices outside $N[\{y_1, \dots, y_p\}, G]$ are adjacent endvertices (by the definition of $z_1, \dots, z_p$), contradicting $\mathbf{P}(G) = \{x_1, \dots, x_p\}$. Thus $\gamma(G) = p$ and Theorem 3.11 leads to $G = S_{p,r}$ as is required.

Case 2. There is a vertex $x \in V(G)$ with $xx_1, xx_2 \in E(G)$.

By Theorem 3.8(iii), $\mathbf{P}(G) = \{x_1, x_2\} \subseteq \mathbf{B}(G)$ and by Proposition 3.6, $x \in \mathbf{Fi}^1(G)$. Since $n \geq 6$, $2 \leq \lfloor n/3 \rfloor$ with equality if and only if $n \in \{6, 7, 8\}$. Note that from Observation 3.2, $\gamma(G) \geq 2$. We shall need the following fact which is an immediately consequence from Theorem 3 [7] (see also [6] pp. 42, Theorem 2.3).

Fact 1. *[7] Let G be a connected graph with $\delta(G) \geq 2$ and $\gamma(G) > 2|V(G)|/5$. If G has a cut-vertex then G is union of two copies of C_4 having exactly one vertex in common.*

Since $\delta(G + x_1x_2) = 2$, by Fact 1 and Theorem 3.3 it follows that $2n/5 \geq \gamma(G + x_1x_2) = \gamma(G)$. Hence $\gamma(G) \leq 3$ and if $n \in \{6, 7\}$ then $\gamma(G) = 2$. First assume $n = 8$ and $\gamma(G) = 3$. Since $\mathbf{V}^-(G) \neq \emptyset$ (Proposition 3.6(1)), x has exactly 3 neighbors. Let $V(G) - \{x, x_1, x_2\} = \{z_1, \dots, z_5\}$ and without loss of generality, $xz_1 \in E(G)$. Denote $Z = \langle \{z_2, \dots, z_5\}, G \rangle$. If $\Delta(Z) = 2$ then $\mathbf{V}^-(G) \neq \emptyset$ - a contradiction (Proposition 3.6(1)). Since G is connected and $\mathbf{P}(G) = \{x_1, x_2\}$, $Z = 2K_2$. But then $\{x, z_1\}$ is a dominating set of G, contradicting $\gamma(G) = 3$.

Thus $n \in \{6, 7, 8\}$ and $\gamma(G) = 2$. Let $u \in V(G) - N[x, G]$. Clearly, $x_1 \in \mathbf{B}(G - u) \cap \mathbf{B}(G)$. Since $x \in \mathbf{Fi}^1(G)$ and $ux \notin E(G)$ it follows that $u \in \mathbf{V}^0(G)$ (Proposition 3.6(1)-(3)). Hence (B_6), applied to u and x_1, holds and by Theorem 2.3(iii) we conclude that $\mathscr{D}(\{u\}, G - x_1) - \mathscr{D}(G) \neq \emptyset$. This implies that $\{u, x_2\}$ is a $\gamma(G + ux_1)$-set. Hence $N[u, G] = V(G) - \{x, x_1, x_2\}$. Assume that $y_1, y_2 \in N(x, G) - \{x_1, x_2\}$ are non adjacent. Hence one of $(B_3) - (B_7)$ holds (applied to y_1 and y_2). Now, by Theorem 2.3 $\mathscr{D}(\{v_1\}, G - v_2) - \mathscr{D}(G) \neq \emptyset$ where $\{v_1, v_2\} = \{y_1, y_2\}$. But $\mathscr{D}(\{v_1\}, G - v_2) = \{\{v_1, x\}\} \subseteq \mathscr{D}(G)$ - a contradiction. Therefore $G - \{x, x_1, x_2\}$ is complete and then $\mathscr{D}(G) = \{\{x, v\} : v \in V(G) - \{x, x_1, x_2\}\} = \mathscr{D}(G + xu)$ whenever $u \in V(G) - N[x, G]$ - a contradiction. □

We recall that a unicyclic graph is a connected graph with a unique cycle

Theorem 3.13. *Let G be a unicyclic graph. Then G is $\#\gamma^+$-EA-critical if and only if $G \in \{C_3, L_{3,1}, L_{4,2}, S_{2,0}\} \cup \{C_{3k+2}, L_{3k+2,3} \ : \ k = 1, 2, \dots\}$.*

Proof. Let $C : y_1, y_2, .., y_r, y_{r+1} = y_1$ be the unique cycle in G. Recall that if G is $\#\gamma^+$-EA-critical then $\mathbf{V}^-(G) = \emptyset$, $\mathbf{Fi}(G) = \mathbf{Fi}^0(G) \cup \mathbf{Fi}^1(G)$, $\langle \mathbf{Fi}^1(G), G\rangle$ is complete (Proposition 3.6(1)-(3)) and if $\mathbf{P}(G) \neq \emptyset$ then $N(\mathbf{P}(G), G) \subseteq \mathbf{Fi}^1(G)$ (by Theorem 3.8(i),(ii) and Proposition 3.6(5)). We let $x \equiv_3 y$ means $x \equiv y (mod\ 3)$.

Case 1. $V(G) = V(C)$.

If $r \equiv_3 1$ then $V(G) = \mathbf{V}^-(G)$. If $r = 3$ then $G = K_3$ is $\#\gamma^+$-EA-critical. If $r \equiv_3 0$ and $r > 3$ then $\{y_i : i \equiv_3 0\}$, $\{y_j : j \equiv_3 1\}$ and $\{y_s : s \equiv_3 2\}$ are the all minimum dominating sets for both G and $G + y_1y_4$. Thus let $r \equiv_3 2$. Note that $V(G) = \mathbf{Fr}^0(G)$ and $V(G - y_k) = \mathbf{G}(G - y_k)$, $k = 1, 2, .., r$. Now if $1 \leq p < q \leq r$ and $y_py_q \notin E(G)$ then each $\gamma(G - y_q)$-set M such that $y_p \in M$ and the endvertices $y_{q-1}, y_{q+1} \notin M$ is a $\gamma(G + y_py_q)$-set which is not a $\gamma(S)$-set.

Case 2. $|\mathbf{P}(G)| \geq 2$ and there is no endvertices at distance two.

Let $\mathbf{P}(G) = \{x_1, x_2, .., x_k\}$. If $k \geq 3$ then since $\langle N(\mathbf{P}(G), G)\rangle$ is complete, G is obtained by adding three paired nonadjacent edges to $K_3 \cup \overline{K_3}$. But then $\mathbf{V}^-(G)$ is not empty.

So, let $k = 2$. Note that $r \not\equiv_3 1$ implies $\{x_1, x_2\} \in \mathbf{V}^-(G)$. If $r = 4$ then $G = S_{2,0}$ which is $\#\gamma^+$-EA-critical (Observation 3.10). Now let $r \geq 7$, $r \equiv_3 1$ and without loss of generality assume $\{y_i\} = N(x_i, G), i = 1, 2$. Then $\{y_1\} \cup \{y_m : m \equiv_3 2\}$ is the unique minimum dominating set of both G and $G + y_ry_3$.

Case 3. $G = L_{r,s}$.

Let $V(G) = V(C) \cup \{x_1, .., x_s\}$ and $E(G) = E(C) \cup \{x_1x_2, x_2x_3, .., x_{s-1}x_s, x_sy_1\}$. It is easy to see that

$$\gamma(L_{r,s}) = \begin{cases} \lfloor (r+s+1)/3 \rfloor & \text{if } r \equiv_3 0 \\ \lceil (r+s)/3 \rceil & \text{otherwise.} \end{cases}$$

Subcase 3.1 $r \equiv_3 0$ and $s \equiv_3 1$.

If $G = L_{3,1}$ then G is $\#\gamma^+$-EA-critical. So, let $G \neq L_{3,1}$. But then $\mathbf{Fi}^1(G) = \{x_i : i \equiv_3 2\} \cup \{y_j : j \equiv_3 1\}$ is the unique $\gamma(G)$ set. Since $\mathbf{Fi}^1(G)$ is independent, G is not $\#\gamma^+$-EA-critical.

Subcase 3.2. $r \equiv_3 0$ and $s \equiv_3 0$.

It is easy to see, that one of the following holds: (a) $s > 3$ and (B_5), applied to $x_3, x_6 \in \mathbf{B}(G)$ holds; hence Theorem 2.3 (i) implies $\mathscr{D}(G) = \mathscr{D}(G + x_3x_6)$, (b) $s = 3$, $r = 3$ and $\mathscr{D}(G) = \mathscr{D}(G + x_2y_1)$, and (c) $s = 3$, $r > 3$ and $\mathscr{D}(G) = \mathscr{D}(G + y_1y_4)$.

Subcase 3.3. $r \equiv_3 0$ and $s \equiv_3 2$.

Then $M = \{x_1\} \cup \{y_i : i \equiv_3 1\} \cup \{x_j : j \equiv_3 0\}$ is a $\gamma(G)$-set and $M - \{x_1\}$ is a $\gamma(G - x_1)$-set. Hence $x_1 \in \mathbf{V}^-(G)$.

Subcase 3.4. $r \equiv_3 1$ and $s \not\equiv_3 2$.

Then $M = \{y_{r-1}\} \cup \{y_i : i \equiv_3 1 \text{ and } i < r\} \cup \{x_j : j \equiv_3 2\}$ is a $\gamma(G)$-set and $M - \{y_{r-1}\}$ is a $\gamma(G - y_{r-1})$-set. Hence $y_{r-1} \in \mathbf{V}^-(G)$.

Subcase 3.5. $r \equiv_3 1$ and $s \equiv_3 2$.

If $G = L_{4,2}$ then G is $\#\gamma^+$-EA-critical, so let $G \neq L_{4,2}$. Note that $\mathbf{Fi}^1(G) = \{x_i : i \equiv_3 2\}$ and $\mathbf{Fi}^0(G) = \{y_j : j \equiv_3 0\}$. If $s > 2$ then $\mathbf{Fi}^1(G)$ is independent. If $s = 2$ and $r > 4$ then $\mathbf{Fi}(G)$ is the unique minimum dominating set for both G and $G + y_3y_6$.

Subcase 3.6. $r \equiv_3 2$ and $s \equiv_3 1$.

If $s = 1$ then $\mathscr{D}(G) = \mathscr{D}(G + y_2y_r)$. If $s > 1$ then $\mathbf{Fi}^1(G) = \{y_1\} \cup \{x_i : i \equiv_3 2\}$ is an independent set.

Subcase 3.7. $r \equiv_3 2$ and $s \equiv_3 2$.

Now $M = \{x_1\} \cup \{y_i : i \equiv_3 1\} \cup \{x_j : j \equiv_3 0\}$ is a $\gamma(G)$-set and $M - \{x_1\}$ is a $\gamma(G - x_1)$-set. Hence $x_1 \in \mathbf{V}^-(G)$.

Subcase 3.8. $r \equiv_3 2$ and $s \equiv_3 0$.

Note that $\mathbf{B}(G) = \{x_i : i \equiv_3 0\}$ and $\mathbf{Fr}^0(G) = V(G) - \mathbf{B}(G)$. If $s > 3$ then (B_5), applied to x_3 and x_6, holds and Theorem 2.3(i) implies $\mathscr{D}(G) = \mathscr{D}(G + x_3x_6)$. So, let $s = 3$. By Theorem 2.3, $\gamma(G + e) = \gamma(G)$ and $\#\gamma(G) \leq \#\gamma(G + e)$ for all $e \in E(\overline{G})$. By Case 1, $\#\gamma(C_r + e) > \#\gamma(C_r)$, where $C_r = \langle V(C), G\rangle$ (for all $e \in E(\overline{C_r})$). Let R be a $\gamma(C_r + e)$-set which is not a $\gamma(C_r)$-set. Then $R \cup \{x_2\}$ is a $\gamma(G + e)$-set which is not a $\gamma(G)$-set (for all $e \in E(\overline{C_r})$). Let M_i be a $\gamma(C_r)$-set with $y_1 \notin M_i$ and $y_i \in M_i$ $i = 2, 3, .., r$. Then: (a) $M_i \cup \{x_1\}$ is a $\gamma(G + y_ix_3)$-set which is not a $\gamma(G)$-set, and (b) $M_i \cup \{x_3\}$ is a $\gamma(G + y_ix_1)$-set which is not a $\gamma(G)$-set. Now, let F_j be a $\gamma(C_r - y_j)$-set with $y_{j-1}, y_{j+1} \notin F_j$, $j = 1, .., r$ and $y_0 = y_r$. Then $|F_j| = \gamma(C_r)$ and $F_j \cup \{x_2\}$ is a $\gamma(G + x_2y_j)$-set which is not a $\gamma(G)$-set. If U is a $\gamma(G)$-set with $y_1 \in U$ then $U \cup \{x_3\}$ is a $\gamma(G + y_1x_1)$-set which is not a $\gamma(G)$-set. Finally, for any $\gamma(C_r)$-set, say Q, $Q \cup \{x_3\}$ is a $\gamma(G + x_3x_1)$-set but it is not a $\gamma(G)$-set. Thus G is $\#\gamma^+$-EA-critical.

Case 4. G has two endvertices at distance two, say x_1 and u.

By Theorem 3.8(iii), $\{x_1, u\} = \mathbf{P}(G)$. Let $V(G) = V(C) \cup \{x_1, .., x_s\} \cup \{u\}$ and $E(G) = E(C) \cup \{x_1x_2, x_2x_3, .., x_{s-1}x_s, x_sy_1\} \cup \{uv\}$, where $v = y_1$ when $s = 1$ and $v = x_2$ otherwise. It is easy to see that

$$\gamma(G) = \begin{cases} \lceil r/3 \rceil + \lceil (s+2)/3 \rceil & \text{if } r \equiv_3 1 \\ \lceil r/3 \rceil + \lceil s/3 \rceil & \text{otherwise.} \end{cases}$$

We sketch the rest of the proof.

If $r \not\equiv_3 1$ and $s \equiv_3 1$, or $r \equiv_3 1$ and $s \equiv_3 2$ then $v \in \mathbf{Fi}^2(G)$.

If $r \equiv_3 1$ and $s \not\equiv_3 2$ then $y_3 \in \mathbf{V}^-(G)$.

If $r \not\equiv_3 1$, $s \equiv_3 2$ and $s \geq 5$ then $x_4 \in \mathbf{V}^-(G)$.

If $r \equiv_3 2$ and $s = 2$ then $y_2 \in \mathbf{V}^-(G)$.

If $r \equiv_3 0$, $r > 3$ and $s = 2$ then $\mathscr{D}(G + y_1y_4) = \mathscr{D}(G)$.

If $r = 3$ and $s = 2$ then obviously G is not $\#\gamma^+$-EA-critical.

If $r \not\equiv_3 1$ and $s \equiv_3 0$ then $N(x_2, G) \subseteq \mathbf{B}(G)$ and $\mathscr{D}(G + x_1x_3) = \mathscr{D}(G)$. □

Denote $\mathtt{A}_1 = \{K_2, K_{1,2}, C_3, L_{4,2}, S_{2,0}\}$, $\mathtt{A}_2 = \{L_{3k+2,3} : k = 1, 2, ..\}$ and $\mathtt{A}_3 = \{C_{3k+2} : k = 1, 2, ..\}$.

Theorem 3.14. *Let each component of a nonconnected graph G be either a nontrivial tree or unicyclic. Then G is $\#\gamma^+$-EA-critical if and only if all components of G are in $\mathtt{A}_1 \cup \mathtt{A}_2 \cup \mathtt{A}_3$ and at most one of them is in $\mathtt{A}_1 \cup \mathtt{A}_2$.*

Proof. We may assume that G has exactly two components, say G_1 and G_2, because of Observation 3.1. By Corollary 3.9 and Theorem 3.13 we have $G_1, G_2 \in \{L_{3,1}\} \cup \mathtt{A}_1 \cup \mathtt{A}_2 \cup \mathtt{A}_3$. Let u be a vertex of degree 2 in $L_{3,1} = G_1$ and $v \in \mathbf{G}(G_2)$. Then one of (B_3) and (B_6), applied to u and v, holds and Theorem 2.3(iii) leads to $\mathscr{D}(G+uv) = \mathscr{D}(G) \cup \mathscr{D}(\{v\}, G-u)$. Since $\mathscr{D}(G_1 - u) = \mathscr{D}(G_1)$, it follows $\mathscr{D}(\{v\}, G - u) \subseteq \mathscr{D}(G)$. Thus $G_1, G_2 \in \mathtt{A}_1 \cup \mathtt{A}_2 \cup \mathtt{A}_3$.

For $i = 1, 2$ choose $x_i \in V(G_i)$, $i = 1, 2$: (a) to be arbitrary if $G_i \in \{K_2, C_3\}$, (b) to belong to $\mathbf{Fi}^1(G_i)$ if $G_i \in \{K_{1,2}, L_{4,2}, S_{2,0}\}$, and (c) to have an endvertex as a neighbor if $G_i \in \mathtt{A}_2$. Note that by Proposition 3.6(3), at most one of G_1 and G_2 is in $\{K_{1,2}, L_{4,2}, S_{2,0}\}$. First let $G_1 \in \{K_{1,2}, L_{4,2}, S_{2,0}\}$ and $G_2 \in \{K_2, C_3\} \cup \mathtt{A}_2$. Then (B_3) holds and Theorem 2.3(iii) implies $\mathscr{D}(G + x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_1\}, G - x_2)$. But $\mathscr{D}(\{x_1\}, G - x_2) = \{A \cup B : A \in \mathscr{D}(\{x_1\}, G_1) = \mathscr{D}(G_1)$ and $B \in \mathscr{D}(G_2 - x_2)\}$ and since clearly $\mathscr{D}(G_2 - x_2) \subseteq \mathscr{D}(G_2)$ it follows that $\mathscr{D}(\{x_1\}, G - x_2) \subseteq \mathscr{D}(G)$. Thus G is not $\#\gamma^+$-EA-critical.

Now let both G_1 and G_2 are in $\{K_2, C_3\} \cup \mathtt{A}_2$. Hence (B_7) holds and Theorem 2.3(iv) implies $\mathscr{D}(G+x_1x_2) = \mathscr{D}(G) \cup \mathscr{D}(\{x_1\}, G-x_2) \cup \mathscr{D}(\{x_2\}, G-x_1)$. But $\mathscr{D}(G_i - x_i) \subseteq \mathscr{D}(G_i)$, $i = 1, 2$ which leads to $\mathscr{D}(\{x_1\}, G - x_2) \subseteq \mathscr{D}(G)$ and $\mathscr{D}(\{x_2\}, G - x_1) \subseteq \mathscr{D}(G)$. Hence G is not $\#\gamma^+$-EA-critical.

Thus, at most one of G_1 and G_2 is in $\mathtt{A}_1 \cup \mathtt{A}_2$. Now, let $G_1 \in \mathtt{A}_1 \cup \mathtt{A}_2 \cup \mathtt{A}_3$ and $G_2 = C_{3k+2}$, $k \geq 1$ with $V(G_2) = \{y_1, .., y_{3k+2}\}$ and $E(G_2) = \{y_1y_2, y_2y_3, .., y_{3k+1}y_{3k+2}, y_{3k+2}y_1\}$. First let $u \in \mathbf{G}(G_1)$, $M_1 \in \mathscr{D}(\{u\}, G_1)$ and $M_2 = \{y_i : i \equiv_3 0\} \cup \{y_{3r+1}\}$. Then $M = M_1 \cup M_2$ is a $\gamma(G + uy_1)$-set which is not a $\gamma(G)$-set. Now, let $u \in \mathbf{B}(G_1)$. Since (B_6), applied to u and y_1, holds, Theorem 2.3(iii) implies $\mathscr{D}(G + uy_1) = \mathscr{D}(G) \cup \mathscr{D}(\{y_1\}, G - u)$. Since clearly $\mathscr{D}(G_1 - u) \supsetneq \mathscr{D}(G_1)$, it follows that $\mathscr{D}(\{y_1\}, G - u) \supsetneq \mathscr{D}(G)$. Thus G is $\#\gamma^+$-EA-critical. □

4. Open questions and problems

(1) Characterize/study $\#\gamma^+$-EA-critical graphs G with: (a) $\mathbf{G}(G) = \mathbf{Fi}(G)$; (b) $\mathbf{G}(G) = \mathbf{Fi}^1(G)$; (c) $\mathbf{G}(G) = \mathbf{Fi}^0(G)$; (d) $V(G) = \mathbf{Fr}^0(G)$, and (e) $\gamma(G) = 2$.

(2) If the n-vertex graph G has no isolated vertex then $\gamma(G) \leq \lfloor n/2 \rfloor$ (Ore [8]). A characterization of n-vertex isolate-free connected extremal graphs G of this inequality is obtained independently by Baogen et al. [1] and Randerath and Volkman [10]. It is easy to see that for every such a extremal graph G, either $\mathbf{V}^-(G)$ is not empty or $G \in$

$\{K_2, K_{1,2}, K_3, C_5\}$. Hence all connected $\#\gamma^+$-EA-critical graphs of order n and $\gamma(G) = \lfloor n/2 \rfloor$ are $K_2, K_{1,2}, K_3$ and C_5 (Proposition 3.6(1)). In this connection it is naturally to ask what is the smallest number k for which there are infinitely many connected $\#\gamma^+$-EA-critical graphs G with $k = \lfloor |V(G)|/2 \rfloor - \gamma(G)$.

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