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Roman domination dot-critical trees

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Abstract

A Roman dominating function on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on G. In this paper, we study trees for which contracting any edge decreases the Roman domination number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$. The *degree* $deg(x)$ of a vertex x denotes the number of neighbors of x in G, and $\Delta(G)$ is the *maximum degree* of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. For notation and graph theory terminology in general we follow [3].

For a graph G, let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1 - 1$ correspondence between the functions $f : V(G) \rightarrow \{0, 1, 2\}$ and the ordered partition $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A function $f = (V_0; V_1; V_2)$ is called a γ_R-function (or $\gamma_R(G)$-function when we want to refer f to G), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$, [1, 5, 6].

Roman domination vertex critical graphs are studied in [2]. A graph G is *Roman domination vertex critical*, or just γ_R-vertex critical, if removing any vertex of G decreases the Roman domination number. We say a vertex v is a *Roman domination critical vertex*, or just a γ_R-critical vertex, if $\gamma_R(G - v) < \gamma_R(G)$. Thus a graph G is γ_R-vertex critical if every vertex of G is γ_R-critical.

For a pair of adjacent vertices x, y in a graph G, we denote by $G.(xy)$ the graph obtained by contracting the edge xy. So, $G.(xy)$ may be viewed as the graph obtained from G by deleting the vertices x and y and appending a new vertex, labeled by (xy), that is adjacent to all the vertices of $G - x - y$ that were originally adjacent to either x or y. We call a graph G *Roman domination dot critical*, or just γ_R-dot critical, if $\gamma_R(G.(xy)) < \gamma_R(G)$ for any two adjacent vertices x, y. If G is γ_R-dot critical and $\gamma_R(G) = k$, then we call G a k-γ_R-dot critical graph. In [4] it was shown that if $e = xy$ is an edge of a graph G, then $\gamma_R(G) - 2 \leq \gamma_R(G.(xy)) \leq \gamma_R(G)$.

We have proved [4] that any γ_R-vertex critical graph is γ_R-dot critical. In [2] it was shown that the complete graph K_2 is the only γ_R-vertex critical tree. In this paper we mainly study γ_R-dot critical trees, and we will demonstrate that there exist a lot of γ_R-dot critical trees. Throughout this paper for an edge $e = uv$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e a pendant edge, u a pendant vertex, and v a support vertex.

2. Some general results

We start with some classes of γ_R-dot critical graphs. The proof is straightforward and is therefore omitted.

Proposition 2.1. *(1) The path P_n is γ_R-dot critical if and only if $n \not\equiv 0$ (mod 3).*
(2) The cycle C_n is γ_R-dot critical if and only if $n \not\equiv 0$ (mod 3).
(3) The complete bipartite graph $K_{m,n}$ is γ_R-dot critical if and only if $m = n = 1$ or $min\{m, n\} \geq 2$.

Now we present some extremal γ_R-dot critical trees. Let H be a tree obtained from a path P_4 by adding a pendant edge to each support vertex of P_4. Let H_1 be a tree obtained from a path $P_6 = v_1v_2v_3v_4v_5v_6$ by adding a pendant edge to v_3. Also let H_2 be a tree obtained from $K_{1,3}$ by subdividing each edge. In the following we characterize all k-γ_R-dot critical trees for $k = 3, 4, 5$. The proof for this result is also straightforward and is therefore omitted.

Lemma 2.2.

1. *A tree T is 3-γ_R-dot critical if and only if $T = P_4$.*
2. *A tree T is 4-γ_R-dot critical if and only if $T \in \{P_5, H\}$.*
3. *A tree T is 5-γ_R-dot critical if and only if $T \in \{P_7, H_1, H_2\}$.*

Proposition 2.3. *Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of a graph H without isolated vertices. If G is a graph formed from H by attaching exactly two leaves v_i and w_i and joining them to u_i for $1 \le i \le n$, then G is γ_R-dot critical.*

Proof. Obviously, the function $f = (V_0; V_1; V_2) = (V(G) - V(H), \emptyset, V(H))$ is a γ_R-function of G and therefore we obtain $\gamma_R(G) = 2n$. Now let $e = xy$ be an edge of G. If $x = u_i$ and $y = v_i$ or $y = w_i$, then $\gamma_R(G.(xy)) = 2n - 1$, since H has no isolated vertices. If $x = u_i$ and $y = u_j$ for $i \ne j$, then $\gamma_R(G.(xy)) = 2n - 2$, and the proof is complete. □

Proposition 2.4. *If a support vertex of a graph G is adjacent to at least three leaves, then G is not γ_R-dot critical.*

Proof. Let u be a support vertex of G which is adjacent to at least three leaves, and let $f = (V_0; V_1; V_2)$ be a γ_R-function of G. This leads to $u \in V_2$. Now let v be a leaf adjacent to u. Since the vertex uv of $G.(uv)$ is adjacent to at least two leaves, for any γ_R-function g of $G.(uv)$, we have $g((uv)) = 2$ (or 0 with leaves 1, if uv is adjacent to exactly two leaves). Consequently, G is not γ_R-dot critical. □

By Proposition 2.4, each support vertex in a γ_R-dot critical tree is adjacent to at most two leaves. Let $\mathcal{F}$ be the class of all trees with the property that each support vertex has degree 3 and is adjacent to exactly two leaves. In order to characterize γ_R-dot critical trees in $\mathcal{F}$ we need the following lemmas.

Lemma 2.5. *Let x be a support vertex of degree 3 in a γ_R-dot critical tree and let x be adjacent to two leaves. If y is the vertex adjacent to x with $deg(y) > 1$, then $deg(y) \ge 3$.*

Proof. Let T be a γ_R-dot critical tree, and let $x \in V(T)$ be a support vertex with two leaves z_1, z_2 adjacent to x. Let y be the vertex adjacent to x with $deg(y) > 1$. Suppose to the contrary, that $deg(y) = 2$. Let $y_1 \ne x$ be adjacent to y. Let f be a γ_R-function for $T.(xz_1)$. If $f((xz_1)) \ne 0$, then $g_1 : V(T) \to \{0, 1, 2\}$ defined by $g_1(x) = 2$, $g_1(z_1) = g_1(z_2) = 0$, and $g_1(u) = f(u)$ if $u \notin \{x, z_1, z_2\}$ is a RDF for T. Thus $\gamma_R(T) \le \gamma_R(T.(xz_1))$, a contradiction to the hypothesis that T is γ_R-dot critical. So $f((xz_1)) = 0$. But then either $f(y) = 2$ or $f(z_2) = 2$. Suppose that $f(y) = 2$ (the proof for $f(z_2) = 2$ is similar). Now $g_2 : V(T) \to \{0, 1, 2\}$ defined by $g_2(x) = 2$, $g_2(y_1) = \max\{1, f(y_1)\}$, and $g_2(u) = 0$ if $u \in N(x)$, and $g_2(u) = f(u)$ if $u \notin N[x] \cup \{y_1\}$, is a RDF for T. Thus $\gamma_R(T) \le \gamma_R(T.(xz_1))$, a contradiction. This implies that $deg(y) \ge 3$. □

Lemma 2.6. *Let x be a vertex of degree greater than one in a tree T such that each vertex of $N(x)$ except at most one is a support vertex adjacent to two leaves. Then T is not γ_R-dot critical.*

Proof. Let x be a vertex in a tree T and $k = deg(x)$. Let $N(x) = \{y_1, y_2, \dots, y_k\}$, where y_i is a support vertex adjacent to two leaves for $2 \le i \le k$. Let z_i, z'_i be the two leaves adjacent to y_i. Suppose to the contrary, that T is γ_R-dot critical. Then $\gamma_R(T.(y_2z_2)) < \gamma_R(T)$. Let f be a γ_R-function for $T.(y_2z_2)$. Since $\gamma_R(T.(y_2z_2)) < \gamma_R(T)$, it is a simple matter to see that $f((y_2z_2)) = 0$, $f(z'_2) = 1$ and $f(x) = 2$. Now we define $g : V(T) \to \{0, 1, 2\}$ by $g(x) = g(z_i) = g(z'_i) = 0$ for $2 \le i \le k$, $g(y_i) = 2$ for $2 \le i \le k$, $g(y_1) = \max\{1, f(y_1)\}$, and $g(u) = f(u)$ if $u \notin N[x] \cup \{z_i, z'_i : 2 \le i \le k\}$. Then g is a RDF for T, a contradiction □

Theorem 2.7. *A tree $T \in \mathcal{F}$ is γ_R-dot critical if and only if T is isomorphic to the tree H in Lemma 2.2.*

Proof. Assume to the contrary, that $T \in \mathcal{F} - H$ is a γ_R-dot critical tree. Let $P = v_1v_2 \dots v_k$ be a longest path in T. Since $T \ne H$, we deduce that $k \ge 5$. Thus v_2 is a support vertex which is adjacent to two leaves. By Lemma 2.5, $deg(v_3) \ge 3$. But then each vertex of $N(v_3) \setminus \{v_4\}$ is a support vertex adjacent to two leaves. According to Lemma 2.6, T is not γ_R-dot critical, a contradiction. □

Next we investigate some special trees.

Theorem 2.8. *Let T be a tree consisting of a path $P = v_1v_2 \dots v_n$ such that v_j is adjacent to a leaf u_j for $j \in \{2, 3, ..., n-1\}$. Then T is γ_R-dot critical if and only if*

(1) $n = 3t$ with $t \ge 2$ and $j = 3i$ for $1 \le i \le t-1$ or $j = 3i-2$ for $2 \le i \le t$.

(2) $n = 3t+1$ with $t \ge 2$ and $j = 3i+1$ for $1 \le i \le t-1$.

Proof. First let $n = 3t$ with $t \ge 1$. If $t \ge 2$ and $j = 3i$ for $1 \le i \le t-1$, then it is a simple matter to obtain $\gamma_R(T) = 2t+1$ and $\gamma_R(T.(xy)) = 2t$ for every edge xy of T, and thus T is γ_R-dot critical. For reason of symmetry, T is also γ_R-dot critical in the case that $j = 3i-2$ for $2 \le i \le t$. However, if $j = 3i-1$, then $\gamma_R(T) = 2t$ and $\gamma_R(T.(v_1v_2)) = 2t$, and thus T is not γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 1$. If $j = 3i+1$ for $1 \le i \le t-1$, then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(xy)) = 2t+1$ for every edge xy of T, and thus T is γ_R-dot critical. However, if $j = 3i$ for $1 \le i \le t$, then $\gamma_R(T) = 2t+1$ and $\gamma_R(T.(uv_{3i})) = 2t+1$, and thus T is not γ_R-dot critical. For reason of symmetry, T is also not γ_R-dot critical in the case that $j = 3i-1$ for $1 \le i \le t$.

Third let $n = 3t+2$ with $t \ge 1$. Then $\gamma_R(T) = 2t$ and $\gamma_R(T.(uv_j)) = 2t$, and thus T is not γ_R-dot critical. This completes the proof. □

Theorem 2.9. *Let T be a tree consisting of a path $P = v_1v_2 \dots v_n$ such that v_j is adjacent to a leaf u and v_{j+1} is adjacent to a leaf w, where $2 \le j \le n-2$. Then T is γ_R-dot critical if and only if*

(1) $j \equiv 2 \pmod 3$ and $n \not\equiv 0 \pmod 3$.

(2) $j \equiv 0 \pmod 3$ and $n \not\equiv 1 \pmod 3$.

(3) $j \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$.

Proof. Assume in the following, without loss of generality, that $j-1 \le n-(j+2)+1 = n-(j+1)$.

(1) First let $n = 3t$ with $t \ge 2$. It is easy to verify that $\gamma_R(T) = 2t+1$ and $\gamma_R(T.(v_{3t-1}v_{3t})) = 2t+1$, and thus T is not γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 1$. We observe that $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(xy)) \le 2t+1$ for every edge xy of T, and thus T is γ_R-dot critical.

Third let $n = 3t+2$ with $t \ge 1$. We observe that $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \le 2t+2$ for every edge xy of T, and thus T is γ_R-dot critical.

(2) First let $n = 3t$ with $t \ge 2$. Then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(xy)) \le 2t+1$ for every edge xy of T, and thus T is γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 2$. Then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(v_{3t}v_{3t+1})) = 2t+2$, and thus T is not γ_R-dot critical.

Third let $n = 3t+2$ with $t \ge 2$. We observe that $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \le 2t+2$ for every edge xy of T, and thus T is γ_R-dot critical.

(3) First let $n = 3t$ with $t \ge 3$. Then $\gamma_R(T) = 2t+1$ and $\gamma_R(T.(v_{3t-1}v_{3t})) = 2t+1$, and thus T is not γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 3$. Then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(v_1v_2)) = 2t+2$, and thus T is not γ_R-dot critical.

Third let $n = 3t+2$ with $t \ge 2$. Then $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \le 2t+2$ for every edge xy of T, and thus T is γ_R-dot critical. □

Theorem 2.10. *Let $P = v_1v_2 \dots v_n$ be a path. Then the corona $T = P \circ K_1$ is γ_R-dot critical if and only if $n \not\equiv 0 \pmod 3$.*

Proof. Let u_i be a leaf of T adjacent to v_i for $1 \le i \le n$.

First let $n = 3t$ with $t \ge 1$. Then $\gamma_R(T) = 4t$ and $\gamma_R(T.(u_2v_2)) = 4t$, and thus T is not γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 0$. Then $\gamma_R(T) = 4t+2$ and $\gamma_R(T.(xy)) = 4t+1$ for every edge xy of T. Thus T is γ_R-dot critical.

Third let $n = 3t+2$ with $t \ge 0$. Then $\gamma_R(T) = 4t+3$ and $\gamma_R(T.(xy)) = 4t+2$ for every edge xy of T. Thus T is γ_R-dot critical, and the proof is complete. □

Theorem 2.11. *Let $P = v_1v_2\dots v_n$ be a path, and let $T' = P \circ K_1$ such that u_i is a leaf of T' adjacent to v_i for $1 \le i \le n$. Then the tree T consisting of T' and a further vertex w adjacent to u_n is γ_R-dot critical if and only if $n \not\equiv 1 \pmod 3$.*

Proof. First let $n = 3t$ with $t \ge 1$. Then $\gamma_R(T) = 4t + 1$ and $\gamma_R(T.(xy)) = 4t$ for every edge xy of T. Thus T is γ_R-dot critical.

Second let $n = 3t+1$ with $t \ge 0$. Then $\gamma_R(T) = 4t+2$ and $\gamma_R(T.(u_nv_n)) = 4t+2$, and thus T is not γ_R-dot critical.

Third let $n = 3t+2$ with $t \ge 0$. Then $\gamma_R(T) = 4t+4$ and $\gamma_R(T.(xy)) = 4t+3$ for every edge xy of T. Thus T is γ_R-dot critical, and the proof is complete. □

3. Construction

In this section we give some ways of constructing γ_R-dot critical trees. We begin with the following trivial observation.

Observation 3.1. *(1) Let x be a vertex of degree at least three in a tree T such that at least three vertices of $N(x)$ are support vertices adjacent to exactly one leaf. Then for each γ_R-function f, $f(x) = 2$.*
(2) Let x be a support vertex in a γ_R-dot critical tree and let x be adjacent to exactly two leaves. If y is adjacent to x and not adjacent to any leaf, then for any γ_R-function f for T, $f(y) \neq 1$.

Lemma 3.2. *Let T be a γ_R-dot critical tree, and let x be a vertex such that for each γ_R-function f, $f(x) = 2$. Let y be a vertex adjacent to x. Then for each γ_R-function g of $T.(xy)$, $g((xy)) = 2$.*

Proof. Let T be a γ_R-dot critical tree, and let x be a vertex such that for any γ_R-function f, $f(x) = 2$. Let y be a vertex adjacent to x. Assume to the contrary, that there is a γ_R-function $g = (V_0; V_1; V_2)$ for $T.(xy)$ such that $g((xy)) \neq 2$. If $\gamma_R(T.(xy)) = \gamma_R(T)-2$, then $g_1 = (V_0\setminus\{(xy)\}; (V_1\setminus\{(xy)\})\cup\{x, y\}; V_2)$ is a RDF for T, a contradiction. So suppose that $\gamma_R(T.(xy)) = \gamma_R(T) - 1$. If $g(xy) = 1$, then $g_2 = (V_0; (V_1 \setminus \{(xy)\}) \cup \{x, y\}; V_2)$ is a RDF for T, a contradiction. It remains to suppose that $g(xy) = 0$. Let $z \in V_2 \cap N_{T.(xy)}((xy))$, and without loss of generality assume that $z \in N_T(x)$. In this case $g_3 = ((V_0 \setminus \{(xy)\}) \cup \{x\}; V_1 \cup \{y\}; V_2)$ is a RDF for T, a contradiction □

Now we consider the following operations. Let T be a γ_R-dot critical tree, and let $x \in V(T)$.

$\mathcal{O}_1$: If for each γ_R-function f, $f(x) = 2$, attach a path yzw and join z to x.

$\mathcal{O}_2$: If for each γ_R-function f, $f(x) = 2$, attach a path $v_1v_2\dots v_{3j+2}$ for some non-negative integer j, and join v_1 to x.

$\mathcal{O}_3$: If for each γ_R-function f, $f(x)=1$, attach a path $v_1v_2\dots v_m$ for some positive integer $m\not\equiv 2 \pmod 3$, and join v_1 to x.

$\mathcal{O}_4$: If for each γ_R-function f, $f(x)=0$, attach a path $v_1v_2\dots v_m$ for some positive integer $m\not\equiv 0 \pmod 3$, and join v_1 to x.

Proposition 3.3. *Let T be a k-γ_R-dot critical tree, and let T' be obtained from T by operation $\mathcal{O}_1$. Then T' is $(k+2)$-γ_R-dot critical.*

Proof. Let T be a k-γ_R-dot critical tree, and let x be a vertex of T such that for each γ_R-function f, $f(x)=2$. Let T' be obtained from T by operation $\mathcal{O}_1$. So T' is obtained from T by attaching a path yzw and joining z to x. Let $f=(V_0;V_1;V_2)$ be a γ_R-function for T. We observe that $f_1=(V_0\cup\{z\};V_1\cup\{y,w\},V_2)$ is a RDF for T'. So $\gamma_R(T')\le\gamma_R(T)+2$. We show that $\gamma_R(T')=\gamma_R(T)+2$. Suppose to the contrary that $\gamma_R(T')\ne\gamma_R(T)+2$. Let g be a γ_R-function for T'. It is obvious that $g(y)+g(z)+g(w)\ge 2$. If $\gamma_R(T')\le\gamma_R(T)$, then $g_1: V(T)\to\{0,1,2\}$ defined by $g_1(x)=\max\{1,g(x)\}$ and $g_1(u)=g(u)$ if $u\ne x$, is a RDF for T with weight less than $\gamma_R(T)$, a contradiction. So suppose that $\gamma_R(T')=\gamma_R(T)+1$. If $g(z)=0$, then $g|_{V(T)}$ is a RDF for T, while if $g(z)=2$ g_1 described above, is a RDF for T, both of which is a contradiction. Thus $\gamma_R(T')=\gamma_R(T)+2$.

Next we show that T' is γ_R-dot critical. Let $e=uv\in E(T')$. If $e\in E(T)$, then $\gamma_R(T.(uv))\le\gamma_R(T)-1$. Using $\gamma_R(T')=\gamma_R(T)+2$, we deduce that

$$\gamma_R(T'.(uv))\le\gamma_R(T)-1+2=\gamma_R(T)+1<\gamma_R(T').$$

It remains to assume that $e\in E(T')\setminus E(T)$. Let $h=(V_0;V_1;V_2)$ be a γ_R-function for T. If $e=xz$, then $h_1=(V_0\cup\{y,w\};V_1;(V_2\setminus\{x\})\cup\{(xz)\})$ is a RDF for $T'.(xz)$ and hence $\gamma_R(T'.(xz))\le\gamma_R(T)<\gamma_R(T')$. So suppose that $e=yz$. This time $h_2=(V_0\cup\{(yz)\};V_1\cup\{w\};V_2)$ is a RDF for $T'.(yz)$ and so again $\gamma_R(T'.(yz))\le\gamma_R(T)+1<\gamma_R(T')$. □

Proposition 3.4. *Let T be a k-γ_R-dot critical tree, and let T' be obtained from T by operation $\mathcal{O}_2$. Then T' is $(k+2j+1)$-γ_R-dot critical.*

Proof. Let T be a k-γ_R-dot critical tree, and let x be a vertex of T such that for each γ_R-function f, $f(x)=2$. Let T' be obtained from T by operation $\mathcal{O}_2$. So T' is obtained from T by attaching a path $P_{3j+2}=v_1v_2\dots v_{3j+2}$ for an integer $j\ge 0$, and joining v_1 to x.

We know that $\gamma_R(P_{3j+2})=2j+2$. It is straightforward to verify that $\gamma_R(T')=\gamma_R(T)+2j+1=\gamma_R(T)+\gamma_R(P_{3j+2})-1$. By Lemma 3.2, we need to show that $\gamma_R(T'.e)<\gamma_R(T')$ for $e\in E(P_{3j+2})\cup\{xv_1\}$. Without loss of generality suppose that $e=xv_1$. It follows that $\gamma_R(T'.e)=\gamma_R(T)+2j$. This completes the proof. □

Proposition 3.5. *Let T be a γ_R-dot critical tree, and let T' be obtained from T by operation $\mathcal{O}_3$. Then T' is γ_R-dot critical.*

Proof. Let T be a k-γ_R-dot critical tree, and let x be a vertex of T such that for any γ_R-function f, $f(x) = 1$. Let T' be obtained from T by operation $\mathcal{O}_3$. So T' is obtained from T by attaching a path $v_1 v_2 \dots v_m$ for an integer m and joining v_1 to x. First assume that $m = 3j + 1$ for some integer $j \geq 0$. Let $f = (V_0; V_1; V_2)$ be a γ_R-function for T. Since $\gamma_R(P_{3j+1}) = 2j + 1$, it is a simple matter to see that $\gamma_R(T') = \gamma_R(T) + 2j + 1 = \gamma_R(T) + \gamma_R(P_{3j+1})$. But both T and P_{3j+1} are γ_R-dot critical. So it is sufficient to show that $\gamma_R(T'.(xv_1)) < \gamma_R(T')$. Now we observe that $\gamma_R(T'.(xv_1)) \leq \gamma_R(T) + \gamma_R(P_{3j}) = \gamma_R(T) + 2j < \gamma_R(T')$. The proof for $m \equiv 0 \pmod 3$ is similar. □

Similarly, the following can be verified.

Proposition 3.6. *Let T be a γ_R-dot critical tree, and let T' be obtained from T by operation $\mathcal{O}_4$. Then T' is γ_R-dot critical.*

Now an induction on the number of operations leads us to the following result.

Theorem 3.7. *Let T' be a tree obtained from a γ_R-dot critical tree T by successive operations $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$, where $\mathcal{D}_i \in \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$ for $1 \leq i \leq m$. Then T is γ_R-dot critical.*

Let $n \geq 3$ be a positive integer, and let T be a tree obtained from $K_{1,n}$ by subdividing each edge of $K_{1,n}$. If x is the central vertex of T, then by Observation 3.1 (1), for each γ_R-function f, we observe that $f(x) = 2$, $f(u) = 0$ if $u \in N(x)$ and $f(u) = 1$ if $u \notin N[x]$. Thus we can apply each operation $\mathcal{O}_i$ on T for $i \in \{1, 2, 3, 4\}$, to produce a further γ_R-dot critical tree.

Problem 3.8. *Characterize all γ_R-dot critical trees.*

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