**Abstract**

A Roman dominating function on a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number $\gamma_R(G)$ of $G$ is the minimum weight of a Roman dominating function on $G$. In this paper, we study trees for which contracting any edge decreases the Roman domination number.

**Keywords:** Domination, roman domination, tree, critical.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n$. We denote the open neighborhood of a vertex $v$ of $G$ by $N_G(v)$, or just $N(v)$, and its closed neighborhood by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The degree $\text{deg}(x)$ of a vertex $x$ denotes the number of neighbors of $x$ in $G$, and $\Delta(G)$ is the maximum degree of $G$. If $S$ is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of $G$ induced by $S$. For notation and graph theory terminology in general we follow [3].

For a graph $G$, let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a 1–1 correspondence between the functions $f : V(G) \rightarrow \{0, 1, 2\}$ and the ordered partition $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$. 
A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a Roman dominating function if every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on $G$. A function $f = (V_0; V_1; V_2)$ is called a $\gamma_R$-function (or $\gamma_R(G)$-function when we want to refer $f$ to $G$), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$, [1, 5, 6].

Roman domination vertex critical graphs are studied in [2]. A graph $G$ is Roman domination vertex critical, or just $\gamma_R$-vertex critical, if removing any vertex of $G$ decreases the Roman domination number. We say a vertex $v$ is a Roman domination critical vertex, or just a $\gamma_R$-critical vertex, if $\gamma_R(G - v) < \gamma_R(G)$. Thus a graph $G$ is $\gamma_R$-vertex critical if every vertex of $G$ is $\gamma_R$-critical.

For a pair of adjacent vertices $x, y$ in a graph $G$, we denote by $G.(xy)$ the graph obtained by contracting the edge $xy$. So, $G.(xy)$ may be viewed as the graph obtained from $G$ by deleting the vertices $x$ and $y$ and appending a new vertex, labeled by $(xy)$, that is adjacent to all the vertices of $G - x - y$ that were originally adjacent to either $x$ or $y$. We call a graph $G$ Roman domination dot critical, or just $\gamma_R$-dot critical, if $\gamma_R(G.(xy)) < \gamma_R(G)$ for any two adjacent vertices $x, y$. If $G$ is $\gamma_R$-dot critical and $\gamma_R(G) = k$, then we call $G$ a $k$-$\gamma_R$-dot critical graph. In [4] it was shown that if $e = xy$ is an edge of a graph $G$, then $\gamma_R(G) - 2 \leq \gamma_R(G.(xy)) \leq \gamma_R(G)$.

We have proved [4] that any $\gamma_R$-vertex critical graph is $\gamma_R$-dot critical. In [2] it was shown that the complete graph $K_2$ is the only $\gamma_R$-vertex critical tree. In this paper we mainly study $\gamma_R$-dot critical trees, and we will demonstrate that there exist a lot of $\gamma_R$-dot critical trees. Throughout this paper for an edge $e = uv$ of a graph $G$ with $\deg(u) = 1$ and $\deg(v) > 1$, we call $e$ a pendant edge, $u$ a pendant vertex, and $v$ a support vertex.

2. Some general results

We start with some classes of $\gamma_R$-dot critical graphs. The proof is straightforward and is therefore omitted.

**Proposition 2.1.** (1) The path $P_n$ is $\gamma_R$-dot critical if and only if $n \not\equiv 0 \pmod{3}$.
(2) The cycle $C_n$ is $\gamma_R$-dot critical if and only if $n \not\equiv 0 \pmod{3}$.
(3) The complete bipartite graph $K_{m,n}$ is $\gamma_R$-dot critical if and only if $m = n = 1$ or $\min\{m,n\} \geq 2$.

Now we present some extremal $\gamma_R$-dot critical trees. Let $H$ be a tree obtained from a path $P_4$ by adding a pendant edge to each support vertex of $P_4$. Let $H_1$ be a tree obtained from a path $P_6 = v_1v_2v_3v_4v_5v_6$ by adding a pendant edge to $v_3$. Also let $H_2$ be a tree obtained from $K_{1,3}$ by subdividing each edge. In the following we characterize all $k$-$\gamma_R$-dot critical trees for $k = 3, 4, 5$. The proof for this result is also straightforward and is therefore omitted.
Lemma 2.2.

1. A tree $T$ is $3\gamma_R$-dot critical if and only if $T = P_4$.

2. A tree $T$ is $4\gamma_R$-dot critical if and only if $T \in \{P_5, H\}$.

3. A tree $T$ is $5\gamma_R$-dot critical if and only if $T \in \{P_7, H_1, H_2\}$.

Proposition 2.3. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of a graph $H$ without isolated vertices. If $G$ is a graph formed from $H$ by attaching exactly two leaves $v_i$ and $w_i$ and joining them to $u_i$ for $1 \leq i \leq n$, then $G$ is $\gamma_R$-dot critical.

Proof. Obviously, the function $f = (V_0; V_1; V_2) = (V(G) - V(H), \emptyset, V(H))$ is a $\gamma_R$-function of $G$ and therefore we obtain $\gamma_R(G) = 2n$. Now let $e = xy$ be an edge of $G$. If $x = u_i$ and $y = v_i$ or $y = w_i$, then $\gamma_R(G.(xy)) = 2n - 1$, since $H$ has no isolated vertices. If $x = u_i$ and $y = u_j$ for $i \neq j$, then $\gamma_R(G.(xy)) = 2n - 2$, and the proof is complete.

Proposition 2.4. If a support vertex of a graph $G$ is adjacent to at least three leaves, then $G$ is not $\gamma_R$-dot critical.

Proof. Let $u$ be a support vertex of $G$ which is adjacent to at least three leaves, and let $f = (V_0; V_1; V_2)$ be a $\gamma_R$-function of $G$. This leads to $u \in V_2$. Now let $v$ be a leaf adjacent to $u$. Since the vertex $uv$ of $G.(uv)$ is adjacent to at least two leaves, for any $\gamma_R$-function $g$ of $G.(uv)$, we have $g((uv)) = 2$ (or 0 with leaves 1, if $uv$ is adjacent to exactly two leaves). Consequently, $G$ is not $\gamma_R$-dot critical.

By Proposition 2.4, each support vertex in a $\gamma_R$-dot critical tree is adjacent to at most two leaves. Let $\mathcal{F}$ be the class of all trees with the property that each support vertex has degree 3 and is adjacent to exactly two leaves. In order to characterize $\gamma_R$-dot critical trees in $\mathcal{F}$ we need the following lemmas.

Lemma 2.5. Let $x$ be a support vertex of degree 3 in a $\gamma_R$-dot critical tree and let $x$ be adjacent to two leaves. If $y$ is the vertex adjacent to $x$ with $\deg(y) > 1$, then $\deg(y) \geq 3$.

Proof. Let $T$ be a $\gamma_R$-dot critical tree, and let $x \in V(T)$ be a support vertex with two leaves $z_1, z_2$ adjacent to $x$. Let $y$ be the vertex adjacent to $x$ with $\deg(y) > 1$. Suppose to the contrary, that $\deg(y) = 2$. Let $y_1 \neq x$ be adjacent to $y$. Let $f$ be a $\gamma_R$-function for $T.(xz_1)$. If $f((xz_1)) \neq 0$, then $g_1 : V(T) \to \{0, 1, 2\}$ defined by $g_1(x) = 2$, $g_1(z_1) = g_1(z_2) = 0$, and $g_1(u) = f(u)$ if $u \notin \{x, z_1, z_2\}$ is a RDF for $T$. Thus $\gamma_R(T) \leq \gamma_R(T.(xz_1))$, a contradiction to the hypothesis that $T$ is $\gamma_R$-dot critical. So $f((xz_1)) = 0$. But then either $f(y) = 2$ or $f(z_2) = 2$. Suppose that $f(y) = 2$ (the proof for $f(z_2) = 2$ is similar). Now $g_2 : V(T) \to \{0, 1, 2\}$ defined by $g_2(x) = 2$, $g_2(y_1) = \max\{1, f(y_1)\}$, and $g_2(u) = 0$ if $u \in N(x)$, and $g_2(u) = f(u)$ if $u \notin N[x] \cup \{y_1\}$, is a RDF for $T$. Thus $\gamma_R(T) \leq \gamma_R(T.(xz_1))$, a contradiction. This implies that $\deg(y) \geq 3$. 


Lemma 2.6. Let $x$ be a vertex of degree greater than one in a tree $T$ such that each vertex of $N(x)$ except at most one is a support vertex adjacent to two leaves. Then $T$ is not $\gamma_R$-dot critical.

Proof. Let $x$ be a vertex in a tree $T$ and $k = \text{deg}(x)$. Let $N(x) = \{y_1, y_2, \ldots, y_k\}$, where $y_i$ is a support vertex adjacent to two leaves for $2 \leq i \leq k$. Let $z_i, z'_i$ be the two leaves adjacent to $y_i$. Suppose to the contrary, that $T$ is $\gamma_R$-dot critical. Then $\gamma_R(T,(y_2z_2)) < \gamma_R(T)$. Let $f$ be a $\gamma_R$-function for $T,(y_2z_2)$. Since $\gamma_R(T,(y_2z_2)) < \gamma_R(T)$, it is a simple matter to see that $f((y_2z_2)) = 0$, $f(z'_2) = 1$ and $f(x) = 2$. Now we define $g : V(T) \to \{0,1,2\}$ by $g(x) = g(z_i) = g(z'_i) = 0$ for $2 \leq i \leq k$, $g(y_i) = 2$ for $2 \leq i \leq k$, $g(y_1) = \max\{1,f(y_1)\}$, and $g(u) = f(u)$ if $u \notin N[x] \cup \{z_i, z'_i : 2 \leq i \leq k\}$. Then $g$ is a RDF for $T$, a contradiction

Theorem 2.7. A tree $T \in F$ is $\gamma_R$-dot critical if and only if $T$ is isomorphic to the tree $H$ in Lemma 2.2.

Proof. Assume to the contrary, that $T \in F - H$ is a $\gamma_R$-dot critical tree. Let $P = v_1v_2 \ldots v_k$ be a longest path in $T$. Since $T \not\cong H$, we deduce that $k \geq 5$. Thus $v_2$ is a support vertex which is adjacent to two leaves. By Lemma 2.5, $\text{deg}(v_3) \geq 3$. But then each vertex of $N(v_3) \setminus \{v_4\}$ is a support vertex adjacent to two leaves. According to Lemma 2.6, $T$ is not $\gamma_R$-dot critical, a contradiction.

Next we investigate some special trees.

Theorem 2.8. Let $T$ be a tree consisting of a path $P = v_1v_2 \ldots v_n$ such that $v_j$ is adjacent to a leaf $u_j$ for $j \in \{2,3,\ldots,n-1\}$. Then $T$ is $\gamma_R$-dot critical if and only if

1. $n = 3t$ with $t \geq 2$ and $j = 3i$ for $1 \leq i \leq t - 1$ or $j = 3i - 2$ for $2 \leq i \leq t$.
2. $n = 3t + 1$ with $t \geq 2$ and $j = 3i + 1$ for $1 \leq i \leq t - 1$.

Proof. First let $n = 3t$ with $t \geq 1$. If $t \geq 2$ and $j = 3i$ for $1 \leq i \leq t - 1$, then it is a simple matter to obtain $\gamma_R(T) = 2t + 1$ and $\gamma_R(T,(xy)) = 2t$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical. For reason of symmetry, $T$ is also $\gamma_R$-dot critical in the case that $j = 3i - 2$ for $2 \leq i \leq t$. However, if $j = 3i - 1$, then $\gamma_R(T) = 2t$ and $\gamma_R(T,(v_1v_2)) = 2t$, and thus $T$ is not $\gamma_R$-dot critical.

Second let $n = 3t + 1$ with $t \geq 1$. If $j = 3i + 1$ for $1 \leq i \leq t - 1$, then $\gamma_R(T) = 2t + 2$ and $\gamma_R(T,(xy)) = 2t + 1$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical. However, if $j = 3i$ for $1 \leq i \leq t$, then $\gamma_R(T) = 2t + 1$ and $\gamma_R(T,(u_3v_i)) = 2t + 1$, and thus $T$ is not $\gamma_R$-dot critical. For reason of symmetry, $T$ is also not $\gamma_R$-dot critical in the case that $j = 3i - 1$ for $1 \leq i \leq t$.

Third let $n = 3t + 2$ with $t \geq 1$. Then $\gamma_R(T) = 2t$ and $\gamma_R(T,(uv_j)) = 2t$, and thus $T$ is not $\gamma_R$-dot critical. This completes the proof.
Theorem 2.9. Let $T$ be a tree consisting of a path $P = v_1v_2 \ldots v_n$ such that $v_j$ is adjacent to a leaf $u$ and $v_{j+1}$ is adjacent to a leaf $w$, where $2 \leq j \leq n-2$. Then $T$ is $\gamma_R$-dot critical if and only if
\begin{enumerate}
\item $j \equiv 2 \pmod{3}$ and $n \not\equiv 0 \pmod{3}$.
\item $j \equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$.
\item $j \equiv 1 \pmod{3}$ and $n \equiv 2 \pmod{3}$.
\end{enumerate}

Proof. Assume in the following, without loss of generality, that $j-1 \leq n - (j+2) + 1 = n - (j+1)$.

(1) First let $n = 3t$ with $t \geq 2$. It is easy to verify that $\gamma_R(T) = 2t + 1$ and $\gamma_R(T.(v_{3t-1}v_{3t})) = 2t + 1$, and thus $T$ is not $\gamma_R$-dot critical.

Second let $n = 3t+1$ with $t \geq 1$. We observe that $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(xy)) \leq 2t+1$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical.

Third let $n = 3t+2$ with $t \geq 1$. We observe that $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \leq 2t+2$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical.

(2) First let $n = 3t$ with $t \geq 2$. Then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(xy)) \leq 2t+1$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical.

Second let $n = 3t + 1$ with $t \geq 2$. Then $\gamma_R(T) = 2t + 2$ and $\gamma_R(T.(v_3v_{3t+1})) = 2t + 2$, and thus $T$ is not $\gamma_R$-dot critical.

Third let $n = 3t+2$ with $t \geq 2$. We observe that $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \leq 2t+2$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical.

(3) First let $n = 3t$ with $t \geq 3$. Then $\gamma_R(T) = 2t+1$ and $\gamma_R(T.(v_{3t-1}v_{3t})) = 2t+1$, and thus $T$ is not $\gamma_R$-dot critical.

Second let $n = 3t+1$ with $t \geq 3$. Then $\gamma_R(T) = 2t+2$ and $\gamma_R(T.(v_1v_2)) = 2t+2$, and thus $T$ is not $\gamma_R$-dot critical.

Third let $n = 3t+2$ with $t \geq 2$. Then $\gamma_R(T) = 2t+3$ and $\gamma_R(T.(xy)) \leq 2t+2$ for every edge $xy$ of $T$, and thus $T$ is $\gamma_R$-dot critical. \qed

Theorem 2.10. Let $P = v_1v_2 \ldots v_n$ be a path. Then the corona $T = P \circ K_1$ is $\gamma_R$-dot critical if and only if $n \not\equiv 0 \pmod{3}$.

Proof. Let $u_i$ be a leaf of $T$ adjacent to $v_i$ for $1 \leq i \leq n$.

First let $n = 3t$ with $t \geq 1$. Then $\gamma_R(T) = 4t$ and $\gamma_R(T.(u_2v_2)) = 4t$, and thus $T$ is not $\gamma_R$-dot critical.

Second let $n = 3t + 1$ with $t \geq 0$. Then $\gamma_R(T) = 4t+2$ and $\gamma_R(T.(xy)) = 4t+1$ for every edge $xy$ of $T$. Thus $T$ is $\gamma_R$-dot critical.

Third let $n = 3t+2$ with $t \geq 0$. Then $\gamma_R(T) = 4t+3$ and $\gamma_R(T.(xy)) = 4t+2$ for every edge $xy$ of $T$. Thus $T$ is $\gamma_R$-dot critical, and the proof is complete. \qed
Theorem 2.11. Let \( P = v_1v_2 \ldots v_n \) be a path, and let \( T' = P \circ K_1 \) such that \( u_i \) is a leaf of \( T' \) adjacent to \( v_i \) for \( 1 \leq i \leq n \). Then the tree \( T \) consisting of \( T' \) and a further vertex \( w \) adjacent to \( u_n \) is \( \gamma_R \)-dot critical if and only if \( n \neq 1 \pmod{3} \).

Proof. First let \( n = 3t \) with \( t \geq 1 \). Then \( \gamma_R(T) = 4t + 1 \) and \( \gamma_R(T.(xy)) = 4t \) for every edge \( xy \) of \( T \). Thus \( T \) is \( \gamma_R \)-dot critical.

Second let \( n = 3t + 1 \) with \( t \geq 0 \). Then \( \gamma_R(T) = 4t + 2 \) and \( \gamma_R(T.(u_nv_n)) = 4t + 2 \), and thus \( T \) is not \( \gamma_R \)-dot critical.

Third let \( n = 3t + 2 \) with \( t \geq 0 \). Then \( \gamma_R(T) = 4t + 4 \) and \( \gamma_R(T.(xy)) = 4t + 3 \) for every edge \( xy \) of \( T \). Thus \( T \) is \( \gamma_R \)-dot critical, and the proof is complete.

\[ \Box \]

3. Construction

In this section we give some ways of constructing \( \gamma_R \)-dot critical trees. We begin with the following trivial observation.

Observation 3.1. (1) Let \( x \) be a vertex of degree at least three in a tree \( T \) such that at least three vertices of \( N(x) \) are support vertices adjacent to exactly one leaf. Then for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \).

(2) Let \( x \) be a support vertex in a \( \gamma_R \)-dot critical tree and let \( x \) be adjacent to exactly two leaves. If \( y \) is adjacent to \( x \) and not adjacent to any leaf, then for any \( \gamma_R \)-function \( f \) for \( T \), \( f(y) \neq 1 \).

Lemma 3.2. Let \( T \) be a \( \gamma_R \)-dot critical tree, and let \( x \) be a vertex such that for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \). Let \( y \) be a vertex adjacent to \( x \). Then for each \( \gamma_R \)-function \( g \) of \( T.(xy) \), \( g((xy)) = 2 \).

Proof. Let \( T \) be a \( \gamma_R \)-dot critical tree, and let \( x \) be a vertex such that for any \( \gamma_R \)-function \( f \), \( f(x) = 2 \). Let \( y \) be a vertex adjacent to \( x \). Assume to the contrary, that there is a \( \gamma_R \)-function \( g = (V_0; V_1; V_2) \) for \( T.(xy) \) such that \( g((xy)) \neq 2 \). If \( \gamma_R(T.(xy)) = \gamma_R(T) - 2 \), then \( g_1 = (V_0 \setminus \{(xy)\}; (V_1 \setminus \{(xy)\}) \cup \{x, y\}; V_2) \) is a RDF for \( T \), a contradiction. So suppose that \( \gamma_R(T.(xy)) = \gamma_R(T) - 1 \). If \( g(xy) = 1 \), then \( g_2 = (V_0; (V_1 \setminus \{(xy)\}) \cup \{x, y\}; V_2) \) is a RDF for \( T \), a contradiction. It remains to suppose that \( g(xy) = 0 \). Let \( z \in V_2 \cap N_T((xy)) \), and without loss of generality assume that \( z \in N_T(x) \). In this case \( g_3 = ((V_0 \setminus \{(xy)\}) \cup \{x\}; V_1 \cup \{y\}; V_2) \) is a RDF for \( T \), a contradiction \( \Box \)

Now we consider the following operations. Let \( T \) be a \( \gamma_R \)-dot critical tree, and let \( x \in V(T) \).

\( O_1 \): If for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \), attach a path \( yzw \) and join \( z \) to \( x \).

\( O_2 \): If for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \), attach a path \( v_1v_2 \ldots v_{3j+2} \) for some non-negative integer \( j \), and join \( v_1 \) to \( x \).
\( O_3: \) If for each \( \gamma_R \)-function \( f \), \( f(x) = 1 \), attach a path \( v_1v_2\ldots v_m \) for some positive integer \( m \not\equiv 2 \pmod{3} \), and join \( v_1 \) to \( x \).

\( O_4: \) If for each \( \gamma_R \)-function \( f \), \( f(x) = 0 \), attach a path \( v_1v_2\ldots v_m \) for some positive integer \( m \not\equiv 0 \pmod{3} \), and join \( v_1 \) to \( x \).

**Proposition 3.3.** Let \( T \) be a \( k-\gamma_R \)-dot critical tree, and let \( T' \) be obtained from \( T \) by operation \( O_1 \). Then \( T' \) is \((k+2)\)-\( \gamma_R \)-dot critical.

**Proof.** Let \( T \) be a \( k-\gamma_R \)-dot critical tree, and let \( x \) be a vertex of \( T \) such that for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \). Let \( T' \) be obtained from \( T \) by operation \( O_1 \). So \( T' \) is obtained from \( T \) by attaching a path \( yzw \) and joining \( z \) to \( x \). Let \( f = (V_0; V_1; V_2) \) be a \( \gamma_R \)-function for \( T \).

We observe that \( f_1 = (V_0 \cup \{z\}; V_1 \cup \{y, w\}, V_2) \) is a RDF for \( T' \). So \( \gamma_R(T') \geq \gamma_R(T) + 2 \).

We show that \( \gamma_R(T') = \gamma_R(T) + 2 \). Suppose to the contrary that \( \gamma_R(T') \neq \gamma_R(T) + 2 \). Let \( g \) be a \( \gamma_R \)-function for \( T' \). It is obvious that \( g(y) + g(z) + g(w) \geq 2 \). If \( \gamma_R(T') \leq \gamma_R(T) \), then \( g_1 : V(T) \rightarrow \{0, 1, 2\} \) defined by \( g_1(x) = \max\{1, g(x)\} \) and \( g_1(u) = g(u) \) if \( u \neq x \), is a RDF for \( T \) with weight less than \( \gamma_R(T) \), a contradiction. So suppose that \( \gamma_R(T') = \gamma_R(T) + 1 \). If \( g(z) = 0 \), then \( g_{|\{T\}} \) is a RDF for \( T \), while if \( g(z) = 2g_1 \) described above, is a RDF for \( T \), both of which is a contradiction. Thus \( \gamma_R(T') = \gamma_R(T) + 2 \).

Next we show that \( T' \) is \( \gamma_R \)-dot critical. Let \( e = uv \in E(T') \). If \( e \in E(T) \), then \( \gamma_R(T.(uv)) \leq \gamma_R(T) - 1 \). Using \( \gamma_R(T') = \gamma_R(T) + 2 \), we deduce that

\[
\gamma_R(T'.(uv)) \leq \gamma_R(T) - 1 + 2 = \gamma_R(T) + 1 < \gamma_R(T').
\]

It remains to assume that \( e \in E(T') \setminus E(T) \). Let \( h = (V_0; V_1; V_2) \) be a \( \gamma_R \)-function for \( T \). If \( e = xz \), then \( h_1 = (V_0 \cup \{y, w\}; V_1; V_0 \setminus \{x\} \cup \{(xz)\}) \) is a RDF for \( T'.(xz) \) and hence \( \gamma_R(T'.(xz)) \leq \gamma_R(T). \) So suppose that \( e = yz \). This time \( h_2 = (V_0 \cup \{(yz)\}; V_1 \cup \{w\}; V_2) \) is a RDF for \( T'.(yz) \) and so again \( \gamma_R(T'.(yz)) \leq \gamma_R(T) + 1 < \gamma_R(T') \). \( \square \)

**Proposition 3.4.** Let \( T \) be a \( k-\gamma_R \)-dot critical tree, and let \( T' \) be obtained from \( T \) by operation \( O_2 \). Then \( T' \) is \((k+2j+1)\)-\( \gamma_R \)-dot critical.

**Proof.** Let \( T \) be a \( k-\gamma_R \)-dot critical tree, and let \( x \) be a vertex of \( T \) such that for each \( \gamma_R \)-function \( f \), \( f(x) = 2 \). Let \( T' \) be obtained from \( T \) by operation \( O_2 \). So \( T' \) is obtained from \( T \) by attaching a path \( P_{3j+2} = v_1v_2\ldots v_{3j+2} \) for an integer \( j \geq 0 \), and joining \( v_1 \) to \( x \).

We know that \( \gamma_R(P_{3j+2}) = 2j + 2 \). It is straightforward to verify that \( \gamma_R(T') = \gamma_R(T) + 2j + 1 = \gamma_R(T) + \gamma_R(P_{3j+2}) - 1 \). By Lemma 3.2, we need to show that \( \gamma_R(T'.e) < \gamma_R(T') \) for \( e \in E(P_{3j+2}) \cup \{xv_1\} \). Without loss of generality suppose that \( e = xv_1 \). It follows that \( \gamma_R(T'.e) = \gamma_R(T) + 2j \). This completes the proof. \( \square \)

**Proposition 3.5.** Let \( T \) be a \( \gamma_R \)-dot critical tree, and let \( T' \) be obtained from \( T \) by operation \( O_3 \). Then \( T' \) is \( \gamma_R \)-dot critical.
Proof. Let \( T \) be a \( k\gamma_R\)-dot critical tree, and let \( x \) be a vertex of \( T \) such that for any \( \gamma_R\)-function \( f \), \( f(x) = 1 \). Let \( T' \) be obtained from \( T \) by operation \( O_3 \). So \( T' \) is obtained from \( T \) by attaching a path \( v_1v_2\ldots v_m \) for an integer \( m \) and joining \( v_1 \) to \( x \). First assume that \( m = 3j + 1 \) for some integer \( j \geq 0 \). Let \( f = (V_0; V_1; V_2) \) be a \( \gamma_R \)-function for \( T \). Since \( \gamma_R(P_{3j+1}) = 2j + 1 \), it is a simple matter to see that \( \gamma_R(T') = \gamma_R(T) + 2j + 1 = \gamma_R(T) + \gamma_R(P_{3j+1}) \). But both \( T \) and \( P_{3j+1} \) are \( \gamma_R \)-dot critical. So it is sufficient to show that \( \gamma_R(T' . (xv_1)) < \gamma_R(T') \). Now we observe that \( \gamma_R(T' . (xv_1)) \leq \gamma_R(T) + \gamma_R(P_{3j}) = \gamma_R(T) + 2j < \gamma_R(T') \). The proof for \( m \equiv 0 \) (mod 3) is similar. \( \square \)

Similarly, the following can be verified.

**Proposition 3.6.** Let \( T \) be a \( \gamma_R \)-dot critical tree, and let \( T' \) be obtained from \( T \) by operation \( O_4 \). Then \( T' \) is \( \gamma_R \)-dot critical.

Now an induction on the number of operations leads us to the following result.

**Theorem 3.7.** Let \( T' \) be a tree obtained from a \( \gamma_R \)-dot critical tree \( T \) by successive operations \( D_1, D_2, \ldots, D_m \), where \( D_i \in \{O_1, O_2, O_3, O_4\} \) for \( 1 \leq i \leq m \). Then \( T \) is \( \gamma_R \)-dot critical.

Let \( n \geq 3 \) be a positive integer, and let \( T \) be a tree obtained from \( K_{1,n} \) by subdividing each edge of \( K_{1,n} \). If \( x \) is the central vertex of \( T \), then by Observation 3.1 (1), for each \( \gamma_R \)-function \( f \), we observe that \( f(x) = 2 \), \( f(u) = 0 \) if \( u \in N(x) \) and \( f(u) = 1 \) if \( u \not\in N[x] \). Thus we can apply each operation \( O_i \) on \( T \) for \( i \in \{1, 2, 3, 4\} \), to produce a further \( \gamma_R \)-dot critical tree.

**Problem 3.8.** Characterize all \( \gamma_R \)-dot critical trees.

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**References**


