AKCE Int. J. Graphs Comb., 8, No. 2 (2011), pp. 105-113

Mean labelings of cyclic snakes

A. Lourdusamy
Department of Mathematics
St.Xavier's College (Autonomous)
Palayamkottai, India.
e-mail: *lourdugnanam@hotmail.com*
and
M. Seenivasan
Department of Mathematics
Sri Paramakalyani College
Alwarkurichi, India.
e-mail: *msvasan_22@yahoo.com*

Communicated by: S. Arumugam
Received 04 November 2010; revised 19 March 2011; accepted 21 July 2011

Abstract

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv depending on the vertex labels. A vertex labeling that assigns for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is called a *mean vertex labeling.* A *mean labeling* f is an injection from V to the set $\{0, 1, 2, \ldots, q\}$ such that the set of edge labels is $\{1, 2, \ldots, q\}$. In this paper we present mean labeling of kC_n-snakes, generalised kC_n-snakes, and super subdivisions of cycles.

Keywords: mean labeling, cyclic snakes, super subdivisions.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv depending on the vertex labels. Let $G = (V, E)$ be a simple graph with p vertices and q edges. A vertex labeling that assigns for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is called a *mean vertex labeling.* A *mean labeling* f is an injection from V to the set $\{0, 1, 2, \ldots, q\}$ such that the set of edge labels is $\{1, 2, \ldots, q\}$. Mean labeling was introduced by Ponraj [3]. Further results on mean labeling are given in the papers [4], [5], [7], [8], [9], and [10]. A graph that has a mean labeling is known as *mean graph.* For a summary on various graph labelings see the dynamic survey of graph labeling by Gallian [2].

Gallian [2] has mentioned that Singh et al. have introduced odd sequential graphs. According to them a graph G with q edges is odd sequential if the vertices can be labeled with distinct integers from the set $\{0, 1, 2, \ldots, q\}$ or in the case of a tree from the set $\{0, 1, 2, \ldots, 2q-1\}$, so that the edge labels induced by addition of the labels of the endpoints take on the values $\{1, 3, \ldots, 2q-1\}$.

Note 1.1. *We observe that an odd sequential graph which is not a tree is a mean graph.*

A kC_n-snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle C_n and the block-cut point graph is a path. Let P be the path of minimum length that contains all the cut vertices of a kC_n-snake. Barrientos [1] has proved that any kC_n-snake is represented by a string $s_1, s_2, \ldots, s_{k-2}$ of integers of length $k-2$ where the i^{th} integer,s_i, on the string is the distance between i^{th} and $(i+1)^{th}$ cut vertices on the path P from one extreme and is taken from $\mathcal{S}_n = \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. The strings obtained for both extremes are assumed to be the same. Then there are at most $\left\lfloor \frac{n}{2} \right\rfloor^{k-2}$ nonisomorphic kC_n-snakes. For example, the string of a $10C_4$-snake shown in Figure 3 is $2, 2, 1, 2, 1, 1, 2, 1$. A kC_n-snake is said to be *linear* if each integer of its string is $\left\lfloor \frac{n}{2} \right\rfloor$.

Sethuraman et al. [6] have defined a super subdivision of a graph as follows: Let G be a graph with n vertices and t edges. A graph H is said to be a *super subdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq t$ in such a way that ends of e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i} after removing the edge e_i from G. Ponraj [3] proved that kC_3-snakes, linear kC_4-snakes and all *super subdivisions* of paths are mean graphs. In this paper we prove that kC_n-snakes for all $k \geq 1$ and $n \geq 3$ are mean graphs and every cycle has a super subdivision which is a mean graph. We also define a generalised kC_n-snake and show that it is a mean graph.

2. Main Results

Theorem 2.1. *The kC_4-snake is odd sequential.*

Proof. Let G be a kC_4-snake with k blocks. As G is bipartite, let one partite set have black vertices and the other have white vertices. Therefore G can be embedded on a square grid as shown in Figure 1.

Consider the following assignment of numbers to the vertices of C_4 as shown in Figure 2 where $x+y$ is an odd integer. Then the induced edge labels defined by the sum of the labels on the end vertices are $x+y, x+y+2, x+y+4, x+y+6$, four successive odd integers starting from $x+y$.

The vertices of G are labeled as follows: The black vertices in the diagonals, except the first vertex of each diagonal, ordered from left to right and inside the diagonals from bottom to top are assigned the numbers from an arithmetic progression of common difference and first term both equal to 4.

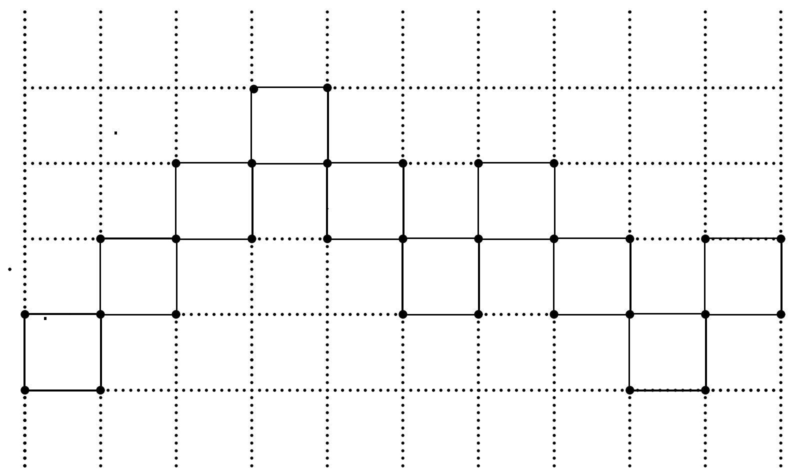


Figure 1: An embedding of $10C_4$-snake

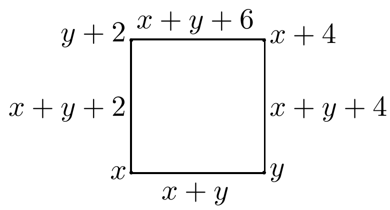


Figure 2: A numbering of a copy of C_4

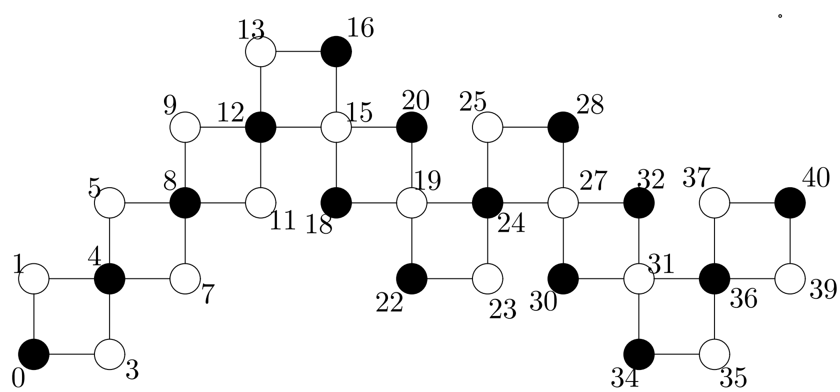


Figure 3: An odd sequential labeling of $10C_4$-snake

The first vertex in the first diagonal is labeled 0 and first vertex in each of the remaining diagonals is assigned the integer which is two more than the label assigned for the last vertex of the previous diagonal. The white vertices, except the first vertex of each diagonal, in the diagonals ordered from left to right and inside the diagonals from top to bottom are assigned the numbers from an arithmetic progression of common difference 4 and first term 3. The first vertex in the first diagonal is labeled 1 and first vertex in each of the remaining diagonals is assigned the integer which is two more than the label assigned for the last vertex of the previous diagonal. An odd sequential labeling of $10C_4$-snake is shown in Figure 3. The labeling for each copy of C_4 is of the same kind that is used in Figure 2 and the black and white vertices are labeled respectively with distinct even and odd integers from $\{0, 1, 2, 3, \ldots, 4k\}$. It can be easily verified that the set of induced edge labels defined by the sum of the labels on the end vertices is the set $\{1, 3, \ldots, 8k - 1\}$. Then G is odd sequential. □

Consider the labelings of C_4 and C_5 given in Figures 4. We call the labeling in (a) and (c) as *type 1* and that in (b) and (d) as *type 2*. In both types of labeling the edge labels induced are $x+1, x+2, x+3$, and $x+4$ on C_4 and $x+1, x+2, x+3, x+4$ and $x+5$ on C_5.

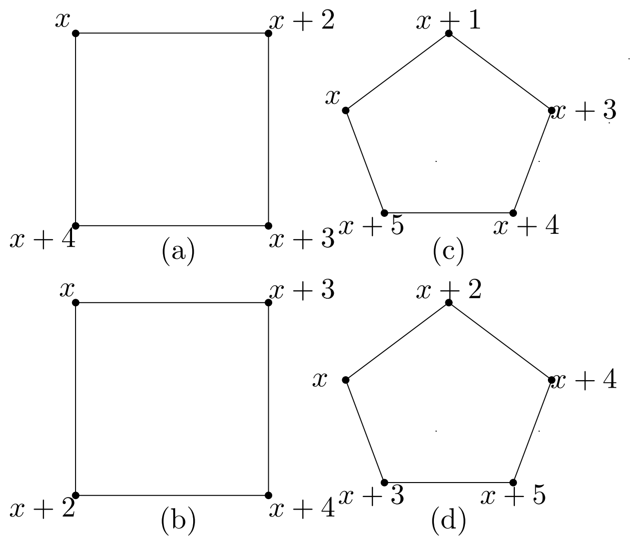


Figure 4: Type 1 and Type 2 labeling of C_4 and C_5

Theorem 2.2. *The kC_n-snake is a mean graph for $n = 4$ and $n = 5$.*

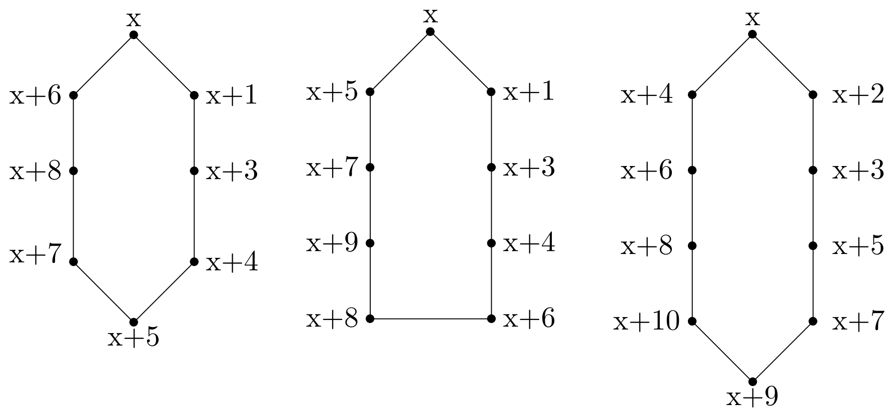
Proof. Let G be any kC_n-snake, for $n = 4$ or $n = 5$. Then its string is of the form $s_1, s_2, s_3, \ldots, s_{k-2}, s_i \in \{1, 2\}$. Let $B_1, B_2, B_3, \ldots, B_k$ be the consecutive blocks of G. First we label the blocks using *type 1* or *type 2* labeling as given below.

Blocks	Type used	Value of x
B_1	either type	0
B_{i+1}	type s_i	ni
B_k	either type	$nk-n$

Then the edge labels induced on the block B_i are $n(i-1)+1, n(i-1)+2, \ldots, ni$. Therefore the edge labels induced on G are $1, 2, 3, \ldots, nk$. Finally B_{i+1} and B_i are connected by identifying their vertices of common label $n(i+1)$ to obtain a mean labeling of G. □

Next we proceed to prove the general case by introducing a notation.

Notation 2.3. *A type d labeling is an injective function $h : V(C_n) \longrightarrow \{x, x+1, \ldots, x+n\}$ that places the labels x and $x+n$ on the vertices at a distance d apart such that the set of induced edge labels is $\{x+1, x+2, \ldots, x+n\}$. This is illustrated in Figure 5.*



C_8: *type* 2 C_9: *type* 3 C_{10}: *type* 4

Figure 5: *type d* labeling of C_n

Let C_n be a cycle with consecutive vertices $v_0, v_1, v_2, \ldots, v_{n-1}$. Let $s \in \mathcal{S}_n$ and $\zeta = \lceil \frac{n}{2} \rceil - s$. Consider the labeling $g_s : V(C_n) \longrightarrow \{x, x+1, \ldots, x+n\}$ defined by the following three steps:

Step 1: Assign the numbers x and $x+n$ respectively to the vertices v_0 and v_s.

Step 2: For $1 \leq i \leq s-1$, the vertices v_{s-i} and v_{s+i} are assigned respectively the numbers $x+n-2i$ and $x+n-2i+1$.

Step 3: The numbers left unassigned to any vertex in step 1 and step 2 except $x+\zeta$ are arranged as an increasing sequence $\alpha_1, \alpha_2, \ldots, \alpha_{n-2s}$ and α_k is assigned to v_{n-k}.

Clearly the vertices of C_n receive distinct labels and the edge labels induced are $x+1, x+2, \ldots, x+n$. If $x = 0$, the labeling is a mean labeling of C_n. Observe that g_s is a *type s* labeling.

Thus we have proved the following lemma.

Lemma 2.4. *The cycle C_n has type s labeling for each $s \in \mathcal{S}_n$.*

Theorem 2.5. *The kC_n-snake is a mean graph for all $k \geq 1$ and $n \geq 3$.*

Proof. This follows from Lemma 2.4 and the methods developed in Theorem 2.2. A mean labeling of $8C_6$-snake is shown in Figure 6. □

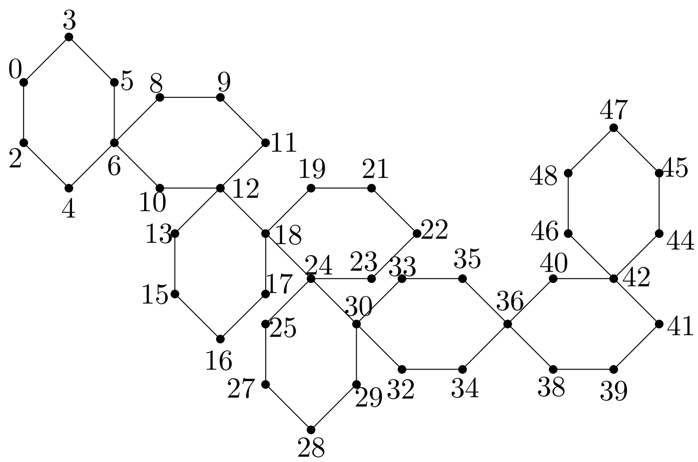


Figure 6: A mean labeling of $8C_6$-snake

Notation 2.6. *A generalised kC_n-snake is defined as a connected graph in which each block is isomorphic to a cycle C_n for some n and the block-cut point graph is a path. It is denoted by $CS(n_1, n_2, n_3, \ldots, n_k)$ where $B_1, B_2, B_3, \ldots, B_k$ are the consecutive blocks and B_i is isomorphic to C_{n_i}. By applying the same methods used to obtain the string of a kC_n-snake, we can show that any generalised kC_n-snake can be represented by a string of integers, $s_1, s_2, s_3, \ldots, s_{k-2}$ of length $k-2$ where $s_{i-1} \in \mathcal{S}_{n_i}$.*

Theorem 2.7. *The generalised kC_n-snake is a mean graph.*

Proof. Let G be any generalised kC_n-snake. Then its string is of the form $s_1, s_2, s_3, \ldots, s_{k-2}, s_{i-1} \in \left\{1, 2, \ldots, \left\lfloor \frac{n_i}{2} \right\rfloor\right\}$. Let $B_1, B_2, B_3, \ldots, B_k$ be the consecutive blocks of G. First we label the blocks as given below:

Blocks	Type used	Value of x
B_1	any type	0
B_{i+1}	*type* s_i	$\sum_1^i n_r$
B_k	any type	$\sum_1^{k-1} n_r$

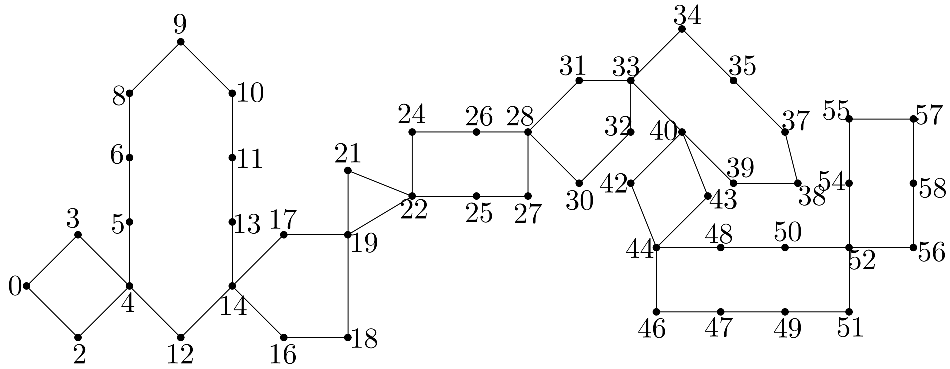


Figure 7: A mean labeling of $CS(10, 4, 5, 3, 6, 5, 7, 4, 8, 6)$

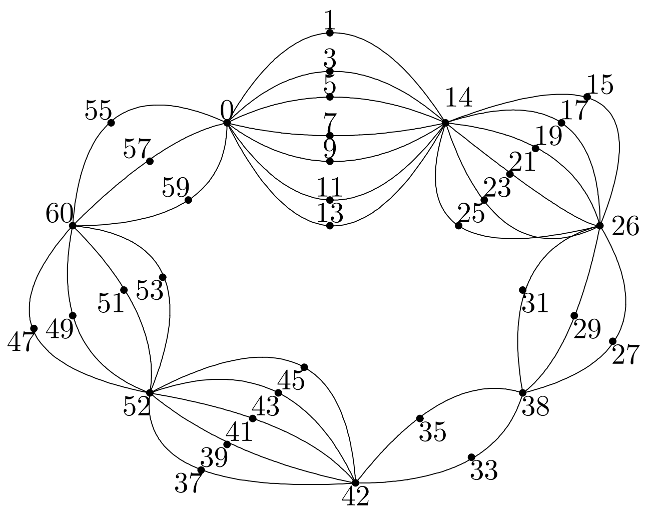


Figure 8: An odd sequential labeling of a super subdivision of C_7

Then the edge labels induced on the block B_i are $\sum_1^{i-1} n_r + 1, \sum_1^{i-1} n_r + 2, \ldots, \sum_1^{i} n_r$. Therefore the edge labels induced on G are $1, 2, 3, \ldots, \sum_1^{i} n_r$. Finally B_{i+1} and B_i are connected by identifying their vertices of common label $\sum_1^i n_r$ to obtain a mean labeling of G. A mean labeling of a $CS(4, 10, 5, 3, 6, 5, 7, 4, 8, 6)$ is shown in Figure 7. □

Lemma 2.8. *For $n \geq 3$ there exist super subdivisions of C_n which are odd sequential.*

Proof. Let C_n be a cycle with consecutive vertices $v_0, v_1, v_2, \ldots, v_{n-1}$. Let $e_i = v_{i-1}v_i$, for $1 \leq i \leq n-1$ and $e_n = v_{n-1}v_0$ be the edges of C_n. Let G be a super subdivision of the cycle C_n obtained by replacing each edge e_i of C_n by a complete bipartite graph K_{2,m_i} where $i \in Z^+$ for $1 \leq i \leq n-1$ such that $m_r = m_n$ and $\sum\limits_{i=1}^{r-1} m_i = \sum\limits_{i=r+1}^{n-1} m_i$ for some r. Let $\{v_{i-1}, v_i\}$ be the two vertices part of K_{2,m_i} for $1 \leq i \leq n-1$ and the two vertices part of K_{2,m_n} be $\{v_{n-1}, v_0\}$. It is clear that G has $n + \sum\limits_{i=1}^{n} m_i$ vertices and $2\sum\limits_{i=1}^{n} m_i$ edges. Let $N_0 = 0, N_{i-1} = N_{i-2} + 2m_i$ for $2 \leq i \leq n$ and $i \neq r+1$ and $N_r = N_{r-1} + 4m_r$. The vertices of G are labelled as follows: Each vertex v_i for $0 \leq i \leq n-1$ is assigned the number N_i which is an even integer. All the vertices in the m_i vertices part of K_{2,m_i} for $1 \leq i \leq n$ ordered by i are put in a string and assigned the consecutive odd integers from 1 to $2\sum\limits_{i=1}^{n} m_i - 1$. An odd sequential labeling of a super subdivision of C_7 is shown in Figure 8. It is clear that the vertices of K_{2,m_i} for $1 \leq i \leq n$ have distinct labels and the induced edge labels defined by the sum of the labels on the end vertices are all distinct positive odd integers.

Therefore, all the vertices of G have distinct labels and the induced edge labels defined by the sum of the labels on the end vertices are distinct odd integers. Hence G is odd sequential. □

Then by Note 1.1, we have the following theorem.

Theorem 2.9. *For $n \geq 3$ there exist super subdivisions of C_n which are mean graphs.*

References

[1] C. Barrientos, Graceful labelings of cyclic snakes, *Ars Combin.*, **60** (2001), 85-96.

[2] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **16** (2009), #DS6.

[3] R. Ponraj, *Studies in labelings of graphs*, Ph.D thesis, Manonmaniam Sundaranar University, India (2004).

[4] R. Ponraj and S. Somasundaram, *Further results on mean graphs*, Proc. SACOEF-ERENCE, National Level Conference, Dr. Sivanthi Aditanar College of Engineering, (2005), 443-448.

[5] D. Ramya, R. Ponraj, and P. Jeyanthi, Super mean labeling of graphs, *Ars Combin.*, (To appear).

[6] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary super subdivisions of graphs, *Indian J.Pure. Appl. Math.*, **32** (2001), 1059-1064.

[7] S. Somasundaram and R. Ponraj, Mean labelings of graphs, *Nat. Acad. Sci. Lett.*, **26** (2003), 210-213.

[8] S. Somasundaram and R. Ponraj, Non-existence of mean labeling for a wheel, *Bull. Pure and Appl. Sciences (Mathematics & Statistics)*, **22E** (2003), 103-111.

[9] S. Somasundaram and R. Ponraj, Some results on mean graphs, *Pure and Applied Mathematical Sciences*, **58** (2003) ,29-35.

[10] S. Somasundaram and R. Ponraj, On mean graphs of order < 5, *J. Decision and Mathematical Sciences*, **9** (2004) 47-58.