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Distributing vertices carefully on a hamiltonian cycle

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Abstract

Given a graph with sufficient minimum degree, the results contained in this work produce a hamiltonian cycle on which specified vertices are placed at approximately prescribed intervals along the cycle. These results improve upon known results in the area by significantly decreasing the error in the approximation when the order of the graph is large.

Keywords: hamiltonian cycle, ordering, distances.

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1. Introduction

For a given integer $t \geq 2$, we will frequently let $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be a set of real numbers with $\sum_{i=1}^{t} \gamma_i = 1$. For a given ordered set $\mathbf{x} = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, we say that G has a $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle if there exists a hamiltonian cycle $H \subseteq G$ with the vertices of $\mathbf{x}$ appearing in order on H with $dist_H(x_i, x_{i+1})$ approximately $\gamma_i n$ where addition is performed modulo t and $n = |G|$. The value of $\epsilon = \max_i (|dist_H(x_i, x_{i+1}) - \gamma_i n|)$ is said to be the *error* of the $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle. We will also use the notation $d_A(v) = |N(v) \cap A|$ and $e(A, B)$ as the number of edges between sets of vertices A and B.

In order to simplify computations, we will frequently assume divisibility of certain numbers. Since the results contained in this work provide structures that approximate a desired structure, such rounding will only minimally affect the outcome. In particular, we may *round* $\boldsymbol{\gamma}$ for n, meaning that we round the individual terms γ_i up or down in order to make $\gamma_i n$ an integer for all i while keeping $\sum_{i=1}^{t} \gamma_i n = n$. Since our goal is to make the error as small as possible, it would be nice if $\epsilon = 0$. If the error $\epsilon = 0$ then we call the cycle a *precise* $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle.

The original motivation for this project is the following result of Kaneko and Yoshimoto [4].

Theorem 1. [4] *Let G be a graph of order n, $d \leq \frac{n}{4}$ a positive integer and A a set of at most $\frac{n}{2d}$ vertices. If $\delta(G) \geq \frac{n}{2}$ then there exists a hamiltonian cycle in G with the distance, along the cycle, between any pair of vertices of A at least d.*

Motivated by Theorem 1 and using our notation, the main result of [3] is stated as follows.

Theorem 2. [3] *Given a set of $t \geq 3$ real numbers $\boldsymbol{\gamma}$ and a constant $\epsilon > 0$, there exists an integer $n_0 \geq \frac{7t^6 \times 10^{10}}{\epsilon^6}$ such that every graph G of order $n \geq n_0$ with $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$, and for any set $\mathbf{x}$ of t vertices, there exists an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most ϵn.*

This result is also related to the following problem attributed to Enomoto [2].

Problem 1. *Given any pair of vertices u and v in a graph G, if $\delta(G) \geq \frac{n+1}{2}$, does there exist a hamiltonian cycle H with $dist_H(u, v) \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

This result would be sharp by the following construction.

Construction 1. *For any odd integer n, consider two copies of $K_{(n+3)/2}$ which share exactly three vertices. If we place both u and v in one of these sets (and not in the shared vertices), there is no hamiltonian cycle with $dist_H(u, v) > \frac{n}{2}$.*

The same idea may be applied for more than 2 chosen vertices. This means that, in general, we cannot hope for a precise result. However, in some cases, almost precision is still possible as we will show.

The first main result of this work is the following improvement upon the error of the hamiltonian cycle in Theorem 2. The proof of Theorem 3 is presented in Section 3.

Theorem 3. *Given a set of $t \geq 3$ real numbers $\boldsymbol{\gamma}$, there exists an integer $n_0 \geq \frac{5t^6 \times 10^{12}}{\gamma_{min}^6}$ (where γ_{min} is the smallest element of $\boldsymbol{\gamma}$) such that every graph G of order $n \geq n_0$ with $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$, and for any set $\mathbf{x}$ of t vertices, there exists an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most some function $f(\boldsymbol{\gamma})$.*

In particular, the error in this result is a constant not depending on $n = |G|$. From the proof, one can show that $f(\boldsymbol{\gamma})$ is on the order of $\frac{t^9}{\gamma_{min}^8}$. In Section 6, we conjecture that this constant can be significantly improved.

More specifically, we also prove the following partial solution to Problem 1. The proof of Theorem 4 is provided in Section 5.

Theorem 4. *Suppose $\boldsymbol{\gamma} = (\frac{1}{2}, \frac{1}{2})$. There exists n_0 such that every graph G of order $n \geq n_0$ with $\delta(G) \geq \frac{n+1}{2}$ and for any set $\mathbf{x}$ of 2 vertices, there exists an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most c where c is an absolute constant.*

From the proof, the value of c is approximately 2.21×10^{12}. The value of n_0 obtained from the proof is much smaller than the value of c so it becomes unnecessary to state.

The proofs of all the above theorems use the following outline. If there is a bipartition of the vertices with very few edges between the parts, we apply a Rebuilding Lemma to construct the desired cycle directly. If not, we first apply a Setup Lemma to create a hamiltonian cycle which is similar to the desired cycle but not quite there. We then modify this cycle using a Swapping Lemma. Since we may lose some vertices in this modification process, we need an Absorbing Lemma to pull lost vertices back into a cycle to make sure it is spanning.

For absorbing vertices into a large cycle, we will apply the following lemma directly from [3].

Lemma 1 (Absorbing [3]). *Let $t \geq 2$, $n \geq 5t$ be integers, and let G be a graph of order n having $\delta(G) \geq \frac{n}{2}$ and let $\mathbf{x} = \{x_1, \ldots, x_t\}$ be an ordered set of t vertices in G. If there exists a cycle C of order at least $\frac{3n}{4} + t$ containing the vertices of $\mathbf{x}$ in the given order, then there exists a hamiltonian cycle H containing the vertices of $\mathbf{x}$ in the given order such that $dist_H(x_i, x_{i+1}) \geq dist_C(x_i, x_{i+1})$ for all $1 \leq i \leq t$.*

In order to get some starting structure at the beginning of the proof, the following setup lemma was used in [3]. In this work, we will actually apply Theorem 2 as a setup lemma with $\epsilon = \frac{\gamma_{min}}{2}$.

Lemma 2 (Setup [3]). *Let $t \geq 3$ be an integer and for sufficiently large n, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. For every ordered set $\mathbf{x} = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing the vertices of $\mathbf{x}$ in order such that $dist_H(x_i, x_{i+1}) \geq \left(\frac{1}{6400t^3(1-\frac{1}{2t})}\right) n$ for all $1 \leq i \leq t$.*

The other two lemmas, proven in Section 2, have to be modified to produce the stronger result. We will use the notation $G[A]$ to denote the subgraph of G induced on the set of vertices A. All other standard notation comes from [1].

Finally, we mention another result which will be used heavily in the proofs. A graph G of order n is called panconnected if between every pair of vertices x and y, there exists a path of each possible length from $dist_G(x, y)$ up to $n - 1$. We will use the following result of Williamson [7] for panconnected graphs.

Theorem 5. [7] *Given a graph G of order n, if $\delta(G) \geq \frac{n}{2} + 1$, then G is panconnected.*

2. Lemmas for Theorem 3

Our first lemma comes almost entirely from [3, 5]. The only significant change from the corresponding swapping lemmas in [3, 5] is a more careful calculation allowing us to strengthen the result. Although the proof is identical until the last two paragraphs, most of it is included for completeness. The proofs of the claims are omitted for the sake of brevity. For a path A from a to a' we use the notation $A[a, a']$.

Lemma 3 (Swapping)**.** *Let $c > 0$ be a constant and G a graph of order n sufficiently large. If $A[a, a']$ and $B[b, b']$ are disjoint paths with $e(A, B) \geq cn^2$, then there exist two other disjoint paths $A^*[a, a'], B^*[b, b'] \subseteq G[A \cup B]$ such that $|B| < |B^*| < |B| + \frac{8c^3+16c^2+1152c+1024}{c^4}$ and $|A^*| + |B^*| \geq |A| + |B| - \frac{64}{c^2}$.*

Proof. Let $A' \subseteq A$ denote the set of vertices $v \in A$ with $d_B(v) \geq \frac{cn^2}{2|A|}$. Since $e(A, B) \geq cn^2$, we find

$$\begin{aligned} |A'| &\geq \frac{cn^2 - (|A| - |A'|)\left(\frac{cn^2}{2|A|}\right)}{|B|} \\ &\geq \frac{cn^2}{2|B|} \\ &\geq \tfrac{c}{2}|A|. \end{aligned}$$

Assign a labelling $l(v)$ to the vertices of A (and B) given by their distance from a (resp. b) along A (resp. B). Define a *crossing pair* to be a pair of edges uy and vz with $u, v \in A$ and $y, z \in B$ such that $l(u) < l(v)$ and $l(z) < l(y)$. We call the vertices $u, v \in A$ of a crossing pair the *base* of the pair and $y, z \in B$ the *terminal vertices* of the pair. Define the *gap* of a crossing pair to be $l(y) - l(z) > 0$. We will concern ourselves only with crossing pairs with gap length at most $\frac{4}{c}$.

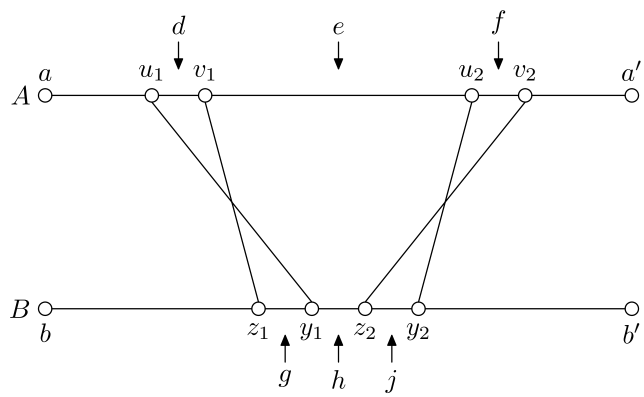

Figure 1: Swapping structure.

Consider Figure 1 consisting of two crossing pairs u_1y_1, v_1z_1 and u_2y_2, v_2z_2 where $d = l(v_1) - l(u_1) > 0$, $e = l(u_2) - l(v_1) > 0$, $f = l(v_2) - l(u_2) > 0$, $g = l(y_1) - l(z_1) > 0$, $h = l(z_2) - l(y_1) \geq 0$ and $j = l(y_2) - l(z_2) > 0$. The goal of this lemma is to find two such crossing pairs with

$$g + h + j < e < cn - (g + h + j). \tag{1}$$

and

$$d + f + g + j \leq \frac{64}{c^2} \tag{2}$$

Within this structure, the new paths $A^* = a, \ldots, u_1, y_1, \ldots, z_2, v_2, \ldots, a'$ and $B^* = b, \ldots, z_1, v_1, \ldots, u_2, y_2, \ldots, b'$ yield the desired pair of paths.

Partition the vertices of A' into collections of $\lceil \frac{4}{c} \rceil$ consecutive (within A' but not necessarily consecutive within A) vertices. For the sake of simplifying computations, we will assume that $\frac{4}{c}$ is an integer. Notice there are at least $\frac{c^2 n}{8}$ collections. Call each such collection a *chunk*.

Claim 1. [3] *Given a chunk C, there are at least $\frac{c|B|}{2}$ crossing pairs based in C, whose corresponding pairs of terminal vertices are pairwise disjoint in B, with gap length at most $\frac{4}{c}$.*

Given two crossing pairs $u_1 y_1, v_1 z_1$ and $u_2 y_2, v_2 z_2$, we say these pairs form a *swapping structure* if $l(u_i) > l(v_j)$ and $l(z_i) > l(y_j)$ for some choice of $i, j \in \{1, 2\}$. For this choice of i and j, define the *gap* of the swapping structure to be $l(z_i) - l(y_j)$ (i.e. the distance in B between the vertices of the crossing pairs).

Claim 2. [3] *Any collection of $\lceil \frac{8}{c} \rceil$ chunks contains a swapping structure with gap h for some $0 \leq h \leq \frac{16}{c^2}$.*

Given a chunk C, define the *span* of C to be the number of vertices $v \in A$ with $l(v_1) \leq l(v) \leq l(v_2)$ for some $v_1, v_2 \in C$. Some of the following inequalities will be strict so, for the sake of simplicity, we will ignore floors and ceilings on fractions for the remainder of this proof. Since the chunks are subsets of A' of order $\frac{4}{c}$ and $|A'| \geq \frac{c}{2}|A|$, the average span of the chunks is at most $\frac{8}{c^2}$. Recall that there are at least $\frac{c^2}{8}|A|$ chunks. From this, we see that the number of chunks of span at most $\frac{16}{c^2}$ is at least $\frac{c^2}{16}|A|$. We call such short chunks *good* and since there are many such chunks, we consider only those which are good.

Our goal is now to mark good chunks that are of a particular distance apart within A. Start at the beginning of A (in terms of the original labelling) and mark the first good chunk. Since $g + h + j \leq \left[2\frac{4}{c} + \frac{16}{c^2}\right]$ (recall our goal is Inequality (1)), we skip the next $\frac{16(c+1)}{c^2} > \left[2\frac{4}{c} + \frac{16}{c^2}\right]$ vertices of A. This provides a lower bound on the distance between marked chunks. We then mark the next (complete) good chunk and repeat this process until we have crossed the entire length of A.

Since at most $\frac{8}{c} + 9$ chunks may intersect each skipped segment, there are at least $\frac{c^2|A|}{16((8/c)+9)}$ marked chunks. Since there are so many marked chunks in A, there must exist a segment of A of order at most $\frac{16((8/c)+9)}{c^2} \times \frac{8}{c} = \frac{1024+1152c}{c^4}$ containing at least $\frac{8}{c}$ marked chunks.

By Claim 2, there exists a swapping structure within these marked chunks. This is the desired swapping structure (see Inequality (1)) since, using the notation from Figure 1,

$$\begin{aligned} e &\geq \left\lceil \frac{16(c+1)}{c^2} \right\rceil \\ &> g+h+j \end{aligned}$$

and

$$\begin{aligned} e &\leq \frac{1024+1152c}{c^4} \\ &\leq \frac{8c^3+16c^2+1152c+1024}{c^4} - \left(\frac{16}{c^2}\right) - 2\left(\frac{4}{c}\right) \\ &\leq \frac{8c^3+16c^2+1152c+1024}{c^4} - (g+h+j) \end{aligned}$$

for n sufficiently large. Also recall that we have shown $d+f+g+h \leq \frac{64}{c^2}$ so Inequality (2) is satisfied. This completes the proof of this lemma. □

If there exists a partition of G into two sets with very few edges from one set to the other, the following lemma constructs the desired hamiltonian cycle directly (with almost zero error). This proof is similar, in essence, to that of the corresponding rebuilding lemma of [3] but contains a twist which allows us to almost entirely eliminate the error, making the proof a bit more complicated.

Lemma 4 (Rebuilding)**.** *Let $t \geq 3$ be an integer and let $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be positive real numbers (rounded for n) with $\sum_{i=1}^{t} \gamma_i = 1$. For $n \geq \frac{500t-2\gamma_{min}}{3\gamma_{min}}$, let G be a graph on n vertices having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. If there exists a bipartition of $V(G)$ into sets A and B with $|A|, |B| \geq \frac{\gamma_{min} n}{2}$ and $e(A,B) < \frac{\gamma_{min}^2 n^2}{2500}$ where $\gamma_{min} = \min_i \gamma_i$, then for all $\mathbf{x} \subseteq V(G)$ with $|\mathbf{x}| = t$, there exists an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most* 1.

Note that, with some effort, the upper bound on the number of edges between the sets A and B and the lower bound on n could likely be improved using more careful calculations.

Proof. The outline of the proof is as follows. We will first classify a "bad" set (call it D) of vertices which have too many edges to the opposite set of the bipartition. Within the remaining "good" vertices of A and B, we show that the degrees are extremely high and that even small subsets induce panconnected graphs. This is the structure that we use to construct the hamiltonian cycle.

We must first determine exactly how much *transportation* (short paths from $A \setminus (D_A \cup \mathbf{x})$ to $B \setminus (D_B \cup \mathbf{x})$) between A and B we will actually use in the construction. We then use the panconnectivity to find short paths from the chosen vertices of $\mathbf{x}$ to transportation paths and reserve the vertices of these short paths for possible future use. Consider the set of desired paths (segments of the hamiltonian cycle) between chosen vertices that we need to construct. Some of these segments stay within A or B while some cross between the sets. We use the first segment that we construct within A to pick up all of $D \cap B$ and the first segment within B to pick up all of $D \cap A$. We then construct the remaining segments, being careful to always leave enough vertices in A or B to preserve the panconnectivity mentioned before.

We now begin the formal proof. Let D_A (and similarly D_B) be the set of vertices in A (respectively B) with each vertex of D_A (or D_B) having at least $\frac{\gamma_{min} n}{25}$ edges into B (respectively A) and let $D = D_A \cup D_B$. From the hypotheses of the lemma, we know that $|D_A|, |D_B| < \frac{\gamma_{min}}{100} n$. Next, we show that any reasonably large subset of $A \setminus D_A$ and $B \setminus D_B$ is panconnected.

Claim 3. *For any set $A' \subseteq (A \setminus D_A)$ with $|A'| \geq \frac{\gamma_{min}}{4} n$, $G[A']$ is panconnected. For any set $B' \subseteq (B \setminus D_B)$ with $|B'| \geq \frac{\gamma_{min}}{4} n$, $G[B']$ is panconnected.*

Proof of Claim 3: Since $\delta(G[A \setminus D_A]) \geq \frac{n}{2} - \frac{\gamma_{min}}{100} n - \frac{\gamma_{min}}{25} n = \frac{n}{2} - \frac{\gamma_{min}}{20} n$ and symmetrically, $\delta(G[B \setminus D_B]) \geq \frac{n}{2} - \frac{\gamma_{min}}{20} n$, so we see that $|A|, |B| \leq \frac{n}{2} + \frac{\gamma_{min}}{20} n$. Hence:

$$\begin{aligned}
\delta(G[A \setminus D_A]) &\geq \tfrac{n}{2} - \tfrac{\gamma_{min}}{20} n \\
&= \tfrac{|A|+|B|}{2} - \tfrac{\gamma_{min}}{20} n \\
&\geq \tfrac{|A|+|A|-\frac{\gamma_{min}}{10} n}{2} - \tfrac{\gamma_{min}}{20} n \\
&= |A| - \tfrac{\gamma_{min}}{10} n \\
&\geq |A \setminus D_A| - \tfrac{\gamma_{min}}{10} n.
\end{aligned}$$

Thus given $A' \subseteq A \setminus D_A$ with $|A'| \geq \frac{\gamma_{min}}{5} n + 2$,

$$\begin{aligned}
\delta(A') &\geq |A'| - \tfrac{\gamma_{min}}{10} n \\
&\geq \left(\tfrac{|A'|+2}{2} + \tfrac{\gamma_{min}}{10} n\right) - \tfrac{\gamma_{min}}{10} n \\
&\geq \tfrac{|A'|+2}{2}.
\end{aligned}$$

Hence, by Theorem 5, we know $G[A']$ is panconnected. By symmetry, $G[B']$ is also panconnected. $\square_{Claim\ 3}$

The remainder of the proof of this lemma is broken into cases based on the connectivity.

Case 1. *Suppose $\kappa(G) \geq 5t$.*

We have constructed D to be the set of vertices which is, in some sense "misbehaving". Unfortunately, vertices of $\mathbf{x}$ may be in D. Hence, we choose a system $X' = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\}$ consisting of two distinct vertices for each vertex of $\mathbf{x}$

with $x_iu_i, x_iv_i \in E(G)$ for all i such that $X' \subseteq G \setminus (\mathbf{x} \cup D)$. By the degree conditions, there must exist such a set X'. If we connect v_i to u_{i+1} with paths of appropriate lengths for all i modulo t, we will create the desired hamiltonian cycle.

Our next step is to reserve transportation paths between A and B. In order to do this, we must first determine how much transportation we will need. For each segment with $v_i \in A$ and $u_{i+1} \in B$, we must use a single transportation path between the sets. If both v_i and u_{i+1} are in the same set (suppose A), we do not necessarily need any transportation paths to B unless, by the time we construct this segment, all of A has already been used up in other segments. In this situation, the segment construction would need two transportation paths because it will have to go to B and then come back.

In order to count the necessary transportation, we perform a process which we call *measuring*. This process consists of pretending to construct the desired segments while keeping in mind how much transportation we will need to use. After performing this measuring, the actual construction will be trivial since we will simply follow the exact same process.

First we reserve two distinct neighbors in $B \setminus (D \cup \mathbf{x} \cup X')$ for each vertex of D_A and do the same for each vertex of D_B. Call the reserved sets D'_A and D'_B respectively. These sets of vertices will also avoid the transportation paths in the real construction but for now, we simply consider any such set. String together the vertices of $D'_A \cup D_A$ on a path P_A as short as possible using the panconnectivity of B. This path segment will be used on a path in B. See Figure 2. Similarly connect all of $D'_B \cup D_B$ into a single path P_B.

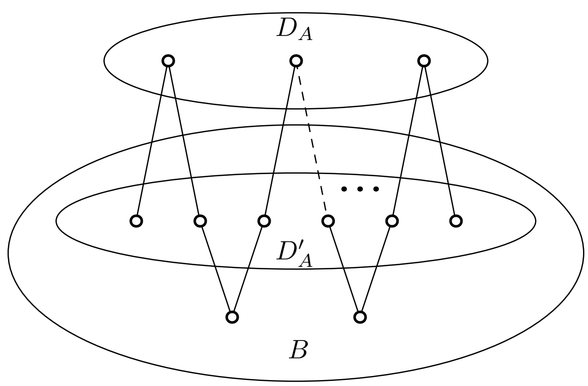

Figure 2: Constructing P_A.

Suppose the first (shortest) path P_j has $v_j \in A$. We use the panconnectivity of A to construct a short path (at most one intermediate vertex) from v_j to the end of P_B. From the other end of P_B, we again use the panconnectivity of A to either construct a path to u_{j+1} of precisely the desired length (if $u_{j+1} \in A$) or we construct a short path (again with at most one intermediate vertex) to a transportation path into B (if $u_{j+1} \in B$). Similarly, the first path which contains an end vertex in B will contain the path P_B (note that if there are no paths with end vertices in one set, say B, we simply use P_A on the first path which uses vertices in B). Since $|D_A|, |D_B| \leq \frac{\gamma_{min} n}{100}$ we know that $|P_A|, |P_B| \leq \frac{\gamma_{min} n}{25} + 1$ which is much smaller than the shortest desired path, so these segments can be easily incorporated into the path construction.

Now suppose we have constructed some segments and suppose the i^{th} segment P_i is the next segment (in increasing order of desired length) that we would like to construct. If v_i and u_{i+1} are both in the same set, say A, and the number of available vertices remaining in A is at least $\gamma_i n + \frac{\gamma_{min} n}{2} + 2(t_A - i)$ (where t_A is the number of vertices of X' left unused in A), we would then simply create the path within A as desired. If the number of available vertices remaining in A is less than $\gamma_i n + \frac{\gamma_{min} n}{2} + 2(t_A - i)$ and at least $\gamma_i n + 2(t-i)$, we have two cases. If there is at least one other vertex of X_i left in A which has not been used, then use the panconnectivity of A to reserve short paths from all but one (call this vertex v) of these remaining vertices to transportation vertices (counting the number needed) and use up any extra vertices of A (not needed for P_i) in a path from v to a transportation vertex. We may then construct P_i using up all the vertices of A with precisely the correct length. If there are no other vertices of X_i left in A, reserve two transportation vertices and create a path between them using all of A except precisely the number of vertices needed for P_i. If the number of available vertices remaining in A is less than $\gamma_i n + 2(t_A - i)$, then first create short paths from each remaining vertex of $X' \cap A$ to transportation vertices, then construct the path P_i by first using up all remaining vertices of A on paths from the ends to transportation vertices, then completing the path of precisely the desired length in B using the panconnectivity of B. Note that this final step may create a path that is a bit longer than $\gamma_i n$ (if $|A|$ is very close to $\gamma_i n + 2(t-i)$) but we may simply go back and reconstruct one of the short paths from $X' \cap A$ to include a few more vertices to avoid this problem.

This completes our measuring process. We may now simply go back through our process to determine exactly the number of transportation paths we need to reserve, call this number k'. Note that, after the mock-construction of each path, we have one of the following:

1. There are at least $\frac{\gamma_{min}}{2} n$ vertices left unused and unreserved (for transportation) in A (or likewise B). This situation allows for easy application of Claim 3.

2. There are very few vertices left unused in A (or likewise B) but all of them are reserved on short paths from our chosen vertices to transportation vertices. In this situation, we no longer need to apply Claim 3 and we are essentially finished using this set.

Since G is $5t$-connected, by Menger's Theorem [6], we know there exists a set of $2t$ vertex disjoint paths from $A \setminus D$ to $B \setminus D$ within $G \setminus (\mathbf{x} \cup X')$. Let M be a collection of k' shortest such paths (see Figure 3). Note that M may use some vertices of D but it will use at most k' vertices in each of $A \setminus D$ and $B \setminus D$. For simplicity sake, redefine $D_A := D_A \setminus V(M)$ and $D_B := D_B \setminus V(M)$.

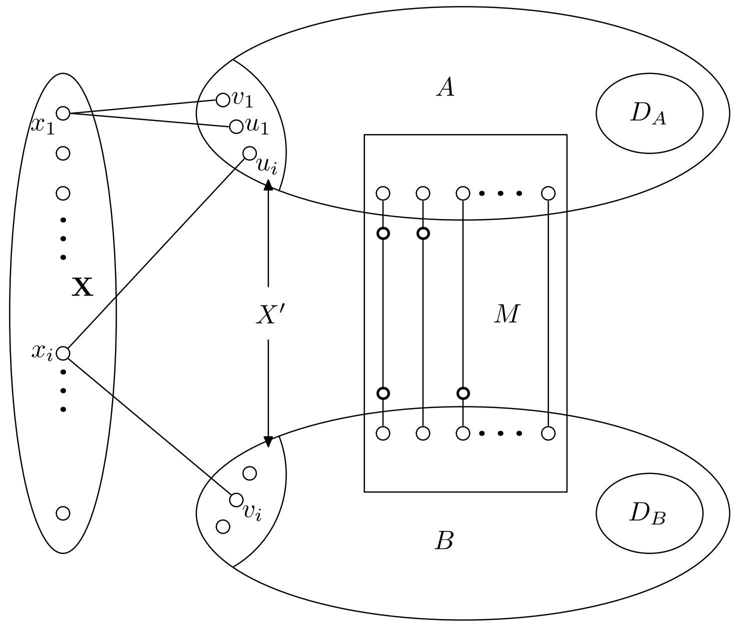

Figure 3: Graph G

Next, we would like to choose a system $D'_A = \{d'_1, d''_1, d'_2, d''_2, \dots\} \subseteq B\setminus(D\cup\mathbf{x}\cup X'\cup M)$ (and likewise $D'_B \subseteq A\setminus(D\cup\mathbf{x}\cup X'\cup M)$) consisting of two distinct vertices in B (respectively A) for each vertex $d_i \in D_A$ with $d_i d'_i \in E(G)$ and $d_i d''_i \in E(G)$. By the definition of D, each vertex of D_A has at least $\frac{\gamma_{min}}{25}n$ edges to B (likewise D_B to A), but there are only at most $\frac{\gamma_{min}}{100}n + 5t$ vertices in $A \cap (D \cup \mathbf{x} \cup X' \cup M)$. Since $\frac{\gamma_{min}}{25}n \geq \frac{\gamma_{min}}{100}n + 5t + 2\frac{\gamma_{min}}{100}$ for $n \geq \frac{500t-2\gamma_{min}}{3\gamma_{min}}$, such a system of representatives must exist.

Finally, we simply follow the process outlined in the measuring to construct the desired paths. One may consider cases based on the relationship between $|A \cup D_B|$ and $|B \cup D_A|$ and the locations of the vertices of $\mathbf{x}$. Since all cases are almost identical and similar to the case presented in the proof of Lemma 5, we omit the case analysis here.

The other cases of the proof are identical to those of the corresponding Rebuilding Lemma in [3]. They are reproduced here for completeness.

Case 2. *Suppose* $\frac{3t}{2} \leq \kappa(G) < 5t$.

Let K be a minimum cutset of G with $\frac{3t}{2} \leq |K| < 5t$. Since $\delta(G) \geq \frac{n}{2}$, there cannot be more than two components of $G \setminus K$. Call these components A and B.

We call a vertex $v \in K$ *blocked to* A (or B) if for every edge e from v into A (respectively B), $e = vx_i$ for some $x_i \in \mathbf{x}$. For each vertex $v \in K \setminus \mathbf{x}$ which is blocked to A, we choose a distinct vertex $x_i \in N(v) \cap \mathbf{x} \cap A$. Call this the *blocking vertex*. We call the vertices of $K \cap \mathbf{x}$ with only one edge to either $A \setminus \mathbf{x}$ (or $B \setminus \mathbf{x}$) *half-blocked* to A (or B).

For $v \in K \setminus \mathbf{x}$ which is blocked by a vertex $x_i \in A \cap \mathbf{x}$, remove all edges to $A \cap \mathbf{x} \setminus x_i$ and move v to B and move x_i to K. By the choice of these removed edges, the connectivity will not be affected. We have now eliminated all the blocked vertices of $K \setminus \mathbf{x}$ and possibly created more half-blocked vertices.

We next remove all edges between vertices of $\mathbf{x}$ to create a new graph G'. Note that these edges are useless in the construction of the desired cycle. Let K' be a minimum cutset in G' containing the maximum number of vertices of $\mathbf{x}$ and observe that we have the following facts about G':

- There are no blocked vertices in K'.
- $\kappa(G') \geq \kappa(G) - \frac{t}{2} \geq t$.
- No half-blocked vertices could also have been blocked or blocking.

For the sake of notation, we distinguish four different types of paths that we would like to construct. A path P_i from x_i to x_{i+1} is of Type I if $x_i \in K'$ and $x_{i+1} \notin K'$ or $x_i \notin K'$ and $x_{i+1} \in K'$. A path P_i is of Type II if $x_i, x_{i+1} \in A$ or both are in B. A path P_i is of Type III if $x_i, x_{i+1} \in K'$. Finally a path P_i is of Type IV if $x_i \in A$ and $x_{i+1} \in B$ or $x_i \in B$ and $x_{i+1} \in A$. See Figure 4.

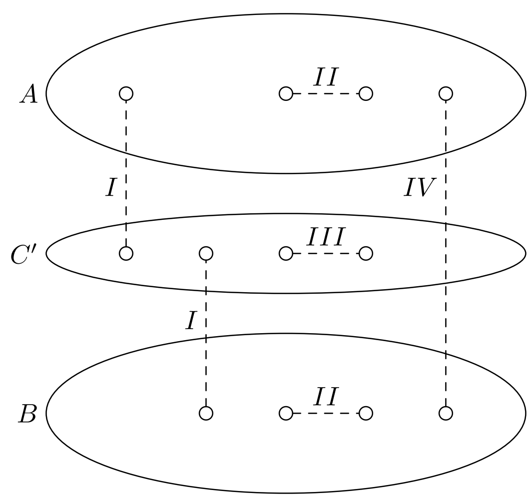

Figure 4: Types of paths.

Since $\delta(G) \geq \frac{n}{2}$ and $|K'| < 5t$, we know $\frac{n}{2} - 5t \leq |A|, |B| \leq \frac{n}{2} + 5t$ and, by Claim 3, $G[A]$ and $G[B]$ are panconnected. Using the same argument as in the previous case, as long as there are enough paths from A to B, we may construct all paths as desired. If $\kappa(G') > t$, the reader may verify that there are enough paths from A to B to complete the above argument. If $\kappa(G') = t$, we know every vertex of $\mathbf{x}$ was either blocked or blocking.

This implies that all the paths are of Types II or IV. By tedious case analysis, the paths may be constructed as in the previous case to get obtain the desired hamiltonian cycle.

Case 3. *Suppose* $\kappa(G) < \frac{3t}{2}$.

Let $k = k_a + k_b$ where k_a is the number of blockings or half-blockings into A and likewise k_b for B. From the previous case, we know that if $\kappa(G) \geq t+1+k$ then we may construct the paths to get the desired hamiltonian cycle. Consider a vertex $v \in A$ and a vertex $w \in B$ which are not involved in any half-blocking. The vertex v is adjacent to at most $\kappa(G) - k_a$ vertices of K and w is adjacent to at most $\kappa(G) - k_b$ vertices of K. Therefore $|A| \geq d(v) + 1 - (\kappa(G) - k_a) \geq \frac{n+t+1}{2} - \kappa(G) + k_a$ and similarly $|B| \geq \frac{n+t+1}{2} - \kappa(G) + k_b$. Hence $n = |A| + |B| + \kappa(G) \geq n + t + 1 - \kappa(G) + k_a + k_b$ or $\kappa(G) \geq t + k_a + k_b + 1$ and we have our result. Note that we can only hope to get the error down to 1 in general since Construction 1 provides an example of where precision is impossible.

This completes the proof of Lemma 4. □

3. Proof of Theorem 3

This proof is very similar to the proof of the main result in [5]. Let us recall the desired statement.

Theorem 3. *Given a set of* $t \geq 3$ *real numbers* $\boldsymbol{\gamma}$, *there exists an integer* $n_0 = \frac{5t^6 \times 10^{12}}{\gamma_{min}^6}$ *such that every graph* G *of order* $n \geq n_0$ *with* $\delta(G) \geq \frac{n+t-1}{2}$ *or* $\delta(G) \geq \frac{n}{2}$ *and* $\kappa(G) \geq \frac{3t}{2}$, *and for any set* $\mathbf{x}$ *of* t *vertices, there exists an* $(\mathbf{x}, \boldsymbol{\gamma})$*-hamiltonian cycle with error at most some function* $f(\boldsymbol{\gamma})$.

Proof. If $V(G)$ can be partitioned into two sets satisfying the hypotheses of Lemma 4, we construct a $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most 3. Therefore, we assume that, for every set $Y \subseteq V(G)$ with $|Y|, |V(G) \setminus Y| \geq \frac{\gamma_{min} n}{2}$, we have

$$e(Y, G \setminus Y) \geq \frac{\gamma_{min}^2 n^2}{2500}. \tag{3}$$

Given an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle H in G, define $f_i = \lceil \gamma_i n \rceil - dist_H(x_i, x_{i+1})$. Let $\mathbf{f} = \{f_1, \ldots f_t\}$ and let $\tilde{\mathbf{f}}$ be the same vector with terms in non-increasing order. Call $\tilde{\mathbf{f}}$ the deficiency vector of H.

Applying Theorem 2 with $\epsilon = \frac{\gamma_{min}}{2}$ guarantees the existence of a hamiltonian cycle containing the vertices of $\mathbf{x}$ in order with at least $\frac{\gamma_{min} n}{2}$ vertices between consecutive chosen vertices. Among all such cycles H, choose the one with the lexicographic least deficiency vector and we would like to show that this H is an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with the error at most $t\frac{8c^3+16c^2+1152c+1024}{c^4}$ where $c = \frac{\gamma_{min}^2}{2500t^2}$. For simplicity, let $g(c) = \frac{8c^3+16c^2+1152c+1024}{c^4}$.

Let $\mathscr{P}$ be the set of paths $P_i[x_i, x_{i+1}]$ between the chosen vertices on the cycle. If all paths in $\mathscr{P}$ are not undersized, then there are not enough vertices left for a path to be oversized. Formally, if $f_1 \leq g(c)$, then $|P_i| \geq \gamma_i n - g(c)$ for all i, which implies $|P_i| < \gamma_i n + tg(c)$ for all i. In other words, if $f_1 \leq g(c)$, then H is the desired $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle.

Let ℓ be the smallest integer such that $f_\ell - f_{\ell+1} > \frac{g(c)}{t}$. Since we may assume $f_1 > g(c)$, it is easy to verify that ℓ exists, with $1 \leq \ell < k$, and that $|P_\ell| < \gamma_\ell n$. Let $Y = V(P_1) \cup \cdots \cup V(P_\ell)$. Since every path has length at least $\frac{\gamma_{min} n}{2}$, we know by (3) that $e(Y, V(G) \setminus Y) \geq \frac{\gamma_{min}^2 n^2}{2500}$. By the pigeon hole principle, there exists a pair of paths $P_B \in \{P_1, \ldots, P_\ell\}$ and $P_A \in \{P_{\ell+1}, \ldots, P_k\}$ with $e(P_A, P_B) \geq \frac{\gamma_{min}^2 n^2}{2500t^2}$.

We may now apply Lemma 3 with $c = \frac{\gamma_{min}^2}{2500t^2}$ to find a pair of paths P_A^* and P_B^*. After replacing P_A and P_B by P_A^* and P_B^* respectively, and applying Lemma 1, we obtain a new $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle. Note that all the path segments of H' between the chosen vertices are still of length at least $\frac{\gamma_{min} n}{2}$. It is also worth noting that P_A^* will not be smaller than P_B or P_B^*, nor will P_B^* be bigger than P_A or P_A^*. This fact follows from the choice of ℓ. Hence, the deficiency vector $\tilde{\mathbf{f}}'$ of H' after application of Lemma 3 and then Lemma 1 is lexicographically smaller than $\tilde{\mathbf{f}}$ since every entry before the ℓ^{th} entry of $\tilde{\mathbf{f}}$ cannot increase. This contradicts the choice of H. Therefore, a $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most $f(\boldsymbol{\gamma})$ exists as desired and the proof is complete. □

4. Lemmas for Theorem 4

As mentioned, the outline of the proof of Theorem 4 is very similar to that of Theorem 3. The Swapping Lemma (Lemma 3) and the Absorbing Lemma (Lemma 1) apply directly and we will use Theorem 1 as a Setup Lemma. Unfortunately, the Rebuilding Lemma (Lemma 4) does not apply in this case because $t = 2$. Therefore we need a separate lemma for this proof.

Lemma 5 (Rebuilding)**.** *Suppose $\boldsymbol{\gamma} = (\frac{1}{2}, \frac{1}{2})$ and $\delta(G) \geq \frac{n+1}{2}$. If there exists a partition of the vertices of a graph G into two parts A and B with $|A|, |B| \geq \frac{n}{4}$ with at most $\frac{n^2}{256}$ edges between the parts, then if n is sufficiently large, for any set $\mathbf{x}$ of 2 vertices, there exists a $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most 1.*

Proof. The outline of this proof is the same as the proof of Lemma 4. Without loss of generality, suppose $|A| \leq |B|$ so $|A| \leq \frac{n}{2}$. Let D_A be the set of vertices in A with at least $\frac{n}{8}$ edges to B. Similarly define D_B and let $D = D_A \cap D_B$. Since there are at most $\frac{n^2}{256}$ edges between A and B, we have $|D_A|, |D_B| \leq \frac{n}{32}$. By the degree condition, we see that

$$|A| \geq \frac{n+1}{2} + 1 - \frac{n^2}{256|A|}.$$

Solving this for $|A|$ yields

$$|A| \geq n\left(\frac{1}{4} - \sqrt{1/16 - 1/256}\right) \geq 0.492n.$$

By the same argument, we also see that $|B| \geq 0.492n$ which means that both sets make up approximately half the graph. We may now show that large subsets of $A \setminus D$ or $B \setminus D$ are panconnected.

Claim 4. *For any subset $A' \subseteq A$ (or similarly $B' \subseteq B$) with $|A'| \geq \frac{n}{4}$ vertices, $G[A']$ is panconnected.*

Proof of Claim 4: First we observe that the minimum degree within $A \setminus D$ is given by

$$\begin{aligned} \delta(G[A \setminus D]) &\geq \frac{n+1}{2} - \frac{n}{8} - |D_A| \\ &\geq \frac{n+1}{2} - \frac{n}{8} - \frac{n}{32} \\ &> |A| - \left(\frac{|A|}{4} - 1\right). \end{aligned}$$

This means that each vertex in A misses fewer than $\frac{|A|}{4} - 1$ vertices of A. Thus, for any set A' with $|A'| \geq \frac{n}{4} \geq \frac{|A|}{2}$, we have $\delta(G[A']) \geq |A'| - \frac{|A|}{4} + 1 \geq \frac{|A'|+2}{2}$ so by Theorem 5, we find that A' is panconnected. $\square_{Claim\ 4}$

We now break the proof into cases based on the connectivity.

Case 1. $\kappa(G) \geq 4$.

As in Lemma 4, we first perform a "measuring" process to determine where to place the paths and how many paths of "transportation" between A and B we need to reserve. Since this process is so similar to the actual construction of the paths, we will simply assume that an appropriate number of transportation paths have been reserved and proceed directly to the construction. Let $\mathscr{M}$ be a shortest collection of such paths from $A \setminus D$ to $B \setminus D$. If any vertices of D are used in this path system, simply redefine D to be $D \setminus V(\mathscr{M})$. Note that, by the degree condition, these paths have length at most 2.

We would like to construct a path (starting and ending in B) through all of D_A. Since each vertex of D_A has at least $\frac{n}{8} > |D_B| + 2|D_A| + 6$ (for n sufficiently large) edges to B, we may choose a set $D'_A \subseteq (B \setminus D_B)$ consisting of two distinct neighbors of v for each $v \in D_A$. Since $B \setminus D_B$ is panconnected, we may connect these vertices of D'_A with at most one intermediate vertex creating a long path containing all of D_A, beginning and ending in B and leaving behind at least $|B \setminus D_B| - 3|D_A| - 1 > \frac{5n}{16}$ vertices in B. Call this path P_A and similarly construct P_B beginning and ending in A.

Now we begin the actual construction of the desired paths. Without loss of generality, suppose $|(A \setminus D_A) \cup D_B| \leq \frac{n}{2}$. We will suppose both chosen vertices are in A. Construct a short path (using the panconnectivity of A) from x_1 to a transportation vertex and two short paths (call them P_1 and P_2) from x_2 to transportation vertices. Also construct a short path from one end of P_{D_B} to the final transportation vertex. In order to ensure that all of these paths disjoint, construct them one at a time, removing the used vertices from the graph.

In what remains of A, construct a spanning path from x_1 to the opposite end (not the one used above) of P_B. This means we now have a path starting at x_1, using all that remained of A, using all of P_B and passing through a transportation path into B. Using the panconnectivity of B, we may then make a path of the desired length to one of the transportation paths back to x_2 to finish our first constructed segment. If this segment is too long, we may make slight adjustments using the lengths of P_1 and P_2.

Finally, our second segment starts at x_2, takes the transportation path back to B, uses all of B and P_A (as above) and ends up crossing the final transportation path back to x_1. This completes the construction of an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most 1. If one or both of the chosen vertices are in B, we may apply a very similar argument to construct a precise $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle.

Case 2. $\kappa(G) = 3$

This means that $\delta(G) = \frac{n+1}{2}$ and the graph is precisely two copies of $K_{(n+3)/2}$ which share three vertices. In this situation, regardless of the placement of x_1 and x_2, it is trivial to construct the desired $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most 1. □

5. Proof of Theorem 4

Recall the desired statement.

Theorem 4 . *Suppose* $\boldsymbol{\gamma} = (\frac{1}{2}, \frac{1}{2})$. *There exists* n_0 *such that every graph* G *of order* $n \geq n_0$ *with* $\delta(G) \geq \frac{n+1}{2}$ *and for any set* $\mathbf{x}$ *of* 2 *vertices, there exists an* $(\mathbf{x}, \boldsymbol{\gamma})$*-hamiltonian cycle with error at most* c *where* c *is an absolute constant.*

Proof. If the vertices of G can be partitioned into sets A and B with $|A|, |B| \geq \frac{n}{4}$ with at most $\frac{n^2}{256}$ edges between the sets, then we apply Lemma 5 to obtain the desired result. Hence we suppose there is no such partition.

By Theorem 1, there exists a hamiltonian cycle with the vertices of $\mathbf{x}$ at distance at least $\frac{n}{4}$ along the cycle. Let H be such a hamiltonian cycle with the greatest distance between the vertices of $\mathbf{x}$. If the distance is at least $\frac{n}{2} - 2.21 \times 10^{12}$, then we have the desired result. Otherwise, apply Lemma 3 with $c = \frac{1}{256}$. We may then use Lemma 1 to reabsorb any vertices lost in the application of Lemma 3. This creates a new hamiltonian cycle H' with $dist_{H'}(x_1, x_2) > dist_H(x_1, x_2)$ since the application of Lemma 3 increases the length of the shorter segment by at most $\frac{4c^3+32c^2+1152c+1024}{c^4} + \frac{64}{c^2} \leq 4.42 \times 10^{12}$. This completes the proof of Theorem 4. □

6. Conclusion

Although Theorem 3 provides an improvement upon the best known result, it would be of great interest to prove the following conjecture.

Conjecture 1. *Given a graph G of order n (with no restriction on n) with $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$, for any set $\mathbf{x}$ of $t \geq 2$ vertices and for any rounded set $\boldsymbol{\gamma}$ of desired lengths, there exists an $(\mathbf{x}, \boldsymbol{\gamma})$-hamiltonian cycle with error at most 1.*

It appears as though the methods used in this work would not suffice to prove this conjecture so an entirely new approach is likely needed.

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