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Outer-k-connected component domination in graphs

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Abstract

A subset S of the vertices of a graph G is an *outer-connected dominating set*, if S is a dominating set of G and $G - S$ is connected. The *outer-connected domination number* of G, denoted by $\widetilde{\gamma}_c(G)$, is the minimum cardinality of an OCDS of G. In this paper we generalize the outer-connected domination in graphs. Many of the known results and bounds of outer-connected domination number are immediate consequences of our results. Let $k \geq 1$ be an integer. A subset D of vertices of G is an outer-k-connected component dominating set if D is a dominating and the graph $G - D$ has exactly k connected component. The outer-k-connected component domination number of G, denoted by $\widetilde{\gamma}_c^k(G)$, is the minimum cardinality of a outer-k-connected component dominating set of G. We study the outer-k-connected component domination in a graph G. We present properties and bounds of outer-k-connected component domination number in graphs, and show that the decision problem for the outer-k-connected component domination number of an arbitrary graph G is NP-complete. Finally, we determine $\widetilde{\gamma}_c^k(G)$ for several certain classes of graphs G.

Keywords: domination; connected; outer-connected domination.

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1. Introduction

For notation and terminology in general we follow [3]. Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$ and size $e = |E(G)|$. We denote the *closed neighborhood* of a vertex v of G by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N[S] = \cup_{v \in S} N[v]$. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. For references in domination see for example [1, 2, 4, 5].

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Cyman in [1] studied the *outer-connected domination* in graphs. A subset S of the vertices of a graph G is an *outer-connected dominating set*, or just OCDS, if S is a dominating set of G and $G-S$ is connected. The *outer-connected domination number* of G, denoted by $\widetilde{\gamma}_c(G)$, is the minimum cardinality of an OCDS of G. Cyman obtained many important results, bounds and characterization regarding this concept.

We generalize the outer-connected domination number of a graph. For an integer k, a subset S of the vertices of a graph G is an *outer-k-connected component dominating set*, or just OkCDS, if S is a dominating set of G and $G-S$ has exactly k components. The *outer-k-connected component domination number* of G, denoted by $\widetilde{\gamma}_c^k(G)$, is the minimum cardinality of an OkCDS of G. In the case that there is no OkCDS of G, we define $\widetilde{\gamma}_c^k(G)=0$. (We remark that we could define $\widetilde{\gamma}_c^k(G)=\infty$ in the case that there is no OkCDS of G). We also refer a $\widetilde{\gamma}_c^k(G)$-set in a graph G as an OkCDS of cardinality $\widetilde{\gamma}_c^k(G)$. Note that an OCDS is an O1CDS, and so in particular many of the known results and bounds of outer-connected domination number are immediate consequences of our results.

In Section 2, we present our general results and bounds for the outer-k-connected component domination number of a graph G. In Section 3, we study the complexity of the outer-k-connected component domination number, and show that the decision problem for the outer-k-connected component domination number of an arbitrary graph G is NP-complete. In section 4, we determine the outer-k-connected component domination number in some special classes of graphs.

All graphs we consider in this paper are without isolated vertex and have at least two vertices. We recall that a leaf in a graph is a vertex of degree one, and a pendant edge is an edge which at least one of its end-points is a leaf.

With K_n we denote the *complete graph* on n vertices, with P_n the *path* on n vertices, with C_n the *cycle* of length n, with W_n the *wheel* with $n+1$ vertices, and with $K_{m,n}$ the *complete bipartite graph* which one partite set has cardinality m and the other partite set has cardinality n. The *corona* $cor(G)$ of a graph G is the graph obtained from G be adding a pendant edge to any vertex of G.

2. General results and bounds

In this section we first obtain some general results for the outer-k-connected component domination number of a graph G, and then we give some bounds for this parameter in terms of some other parameters. All graphs we handle in this section are connected.

Observation 2.1. *If $\widetilde{\gamma}_c^k(G)=0$ for some integer k, then for every $m>k$, $\widetilde{\gamma}_c^m(G)=0$.*

Proof. Let $\widetilde{\gamma}_c^k(G)=0$ for some integer k and $m>k$ be an integer. Suppose to the contrary that $\widetilde{\gamma}_c^m(G)\neq 0$. Let S be a $\widetilde{\gamma}_c^m(G)$-set, and let $A_1,A_2,\ldots,A_m$ be the m components of $G-S$. It is obvious that $S_1=S\cup A_{k+1}\cup\cdots\cup A_m$ is an OkCDS for G and $G-S_1$ has k components. This implies that $\widetilde{\gamma}_c^k(G)>0$, a contradiction. □

Observation 2.2. *Let k be the maximum integer such that $\widetilde{\gamma}_c^k(G) > 0$. If S is an OkCDS, then every connected component of $G - S$ is complete.*

Proof. Let k be the maximum integer such that $\widetilde{\gamma}_c^k(G) > 0$, and let S be an OkCDS. Suppose to the contrary that there is a connected component G_1 of $G \setminus S$ such that G_1 is not complete. Let x, y be two non-adjacent vertices in G_1. Then $S \cup (V(G_1) \setminus \{x, y\})$ is an O$(k+1)$CDS for G, a contradiction. □

Let $\alpha(G)$ be the independence number of G.

Theorem 2.3. *Let $k \geq 1$ be an integer, and let G be a graph without isolated vertices. Then $\tilde{\gamma}_c^k(G) > 0$ if and only if $\alpha(G) \geq k$.*

Proof. Assume first that $\alpha(G) \geq k$. Since $\alpha(G) \geq k$, the graph G contains an independent set I with $|I| = k$. Now define the set $S = V(G) \setminus I$. Since G has no isolated vertices, S is a dominating set of G, and $G - S$ has exactly k components. This shows that $\tilde{\gamma}_c^k(G) > 0$, as desired.

Conversely, assume that $\tilde{\gamma}_c^k(G) > 0$. Let S be a $\tilde{\gamma}_c^k(G)$-set, and let $G_1, G_2, \ldots, G_k$ be the components of $G - S$. If we choose a vertex $x_i \in V(G_i)$ for $i = 1, 2, \ldots, k$, then $I = \{x_1, x_2, \ldots, x_k\}$ is an independent set in G. This implies that $\alpha(G) \geq |I| = k$, and the proof is complete. □

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. *For a graph G of order at least two, $\widetilde{\gamma}_c^2(G) = 0$ if and only if $G = K_n$.*

Next we present some further consequences of Theorem 2.3. The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G.

Corollary 2.5. *Let $k \geq 1$ be an integer, and let G be a graph of order n without isolated vertices. If $\omega(G) \geq n + 2 - k$, then $\tilde{\gamma}_c^k(G) = 0$.*

Proof. Since every independent set of G has at most one vertex of the maximum clique, we observe that $\alpha(G) \leq n - \omega(G) + 1$. Now the hypothesis $\omega(G) \geq n + 2 - k$ implies that

$$\alpha(G) \leq n - \omega(G) + 1 \leq n - n - 2 + k + 1 = k - 1.$$

Using Theorem 1, we conclude that $\tilde{\gamma}_c^k(G) = 0$. □

Corollary 2.6. *Let $k \geq 2$ be an integer, and let G be a graph of order n without isolated vertices. If $\delta(G) \geq n + 1 - k$, then $\tilde{\gamma}_c^k(G) = 0$.*

Proof. Suppose to the contrary that $\tilde{\gamma}_c^k(G) > 0$. Choose a $\tilde{\gamma}_c^k(G)$-set S, and denote by $G_1, G_2, \ldots, G_k$ the components of $G - S$. If v is a vertex of the component G_1, then $d_G(v) \leq n - k$. The hypothesis $\delta(G) \geq n + 1 - k$ leads to the contradiction

$$n + 1 - k \leq \delta(G) \leq d_G(v) \leq n - k.$$

Therefore $\tilde{\gamma}_c^k(G) = 0$, and the proof is complete. □

It is obvious that $\widetilde{\gamma}_c^k(G) \leq n-k$. For $k = 1$ Cyman in [1] characterized all graphs G with $\widetilde{\gamma}_c^1(G) = n-1$. We consider $k > 1$. To characterize graphs achieving equality for the upper bound $\widetilde{\gamma}_c^k(G) \leq n - k$, we need to introduce a family of graphs. For $k > 1$, let $\sum_k$ be the class of all graphs G such that $G \in \sum_k$ if and only if $V(G) = A \cup B$ such that $|A| = n-k$, $G[B] = \overline{K_k}$, and no subset $S \subseteq A \cup B$ with $|S| < n-k$ is an outer-k-connected component dominating set for G. The following is a characterization of graphs G with $\widetilde{\gamma}_c^k(G) = n-k$. The proof is straightforward and is omitted.

Theorem 2.7. *For a graph G of order n, $\widetilde{\gamma}_c^k(G) = n - k$ if and only if $G \in \sum_k$.*

We remark that it is straightforward to see that $\widetilde{\gamma}_c^k(G) = 1$ if and only if G is obtained from the disjoint union of k connected graphs $H_1, H_2, \ldots, H_k$ by adding a new vertex x and joining x to all vertices in $\cup_{i=1}^k H_i$.

The next bound involves the outer-k-connected component domination number of a graph G with the size and the order of graph.

Theorem 2.8. *For a graph G of order n, size e and $\widetilde{\gamma}_c^k(G) > 0$,*

$$\widetilde{\gamma}_c^k(G) \geq n - \lfloor \frac{e+k}{2} \rfloor$$

Proof. Let S be a $\widetilde{\gamma}_c^k(G)$-set of cardinality s, and $G_1, G_2, \ldots, G_k$ be the components of $G - S$. Suppose that $|V(G_i)| = n_i$ for $1 \leq i \leq k$. Since S is a domination set of G, any vertex in G_i has at least one neighbor in S for $1 \leq i \leq k$. On the other hand G_i is a connected and so has at least n_i-1 edges for $i = 1, 2, \ldots, k$. Altogether, we obtain that

$$e \geq \sum_{i=1}^{k}(n_i - 1) + \sum_{i=1}^{k} n_i.$$

Since $\sum_{i=1}^k n_i = n - s$, we have $e \geq 2n - 2s - k$. This implies that $s \geq n - \frac{e+k}{2}$, and the proof is complete. □

Corollary 2.9. [1] *For a graph G of order n and size e,*

$$\widetilde{\gamma}_c(G) \geq n - \frac{e+1}{2}.$$

An immediate consequence of Theorem 2.8 is the following corollary for trees.

Proposition 2.10. *For a tree T of order n and $\widetilde{\gamma}_c^k(G) > 0$,*

$$\widetilde{\gamma}_c^k(T) \geq \lceil \frac{n+1-k}{2} \rceil.$$

Corollary 2.11. [1] *If T is a tree of order $n \geq 3$, then*

$$\widetilde{\gamma}_c(T) \geq \lceil \frac{n}{2} \rceil.$$

We next obtain a sharp upper bound for the outer-k-connected component domination number of a graph G.

Theorem 2.12. *Let G be a graph of order n and size e, and let $k > 1$. If $\widetilde{\gamma}_c^k(G) > 0$, then*

$$\widetilde{\gamma}_c^k(G) \leq n - \sqrt{\frac{k(n^2-n-2e)}{k-1}}.$$

Proof. Assume that $k > 1$, and $\widetilde{\gamma}_c^k(G) > 0$. Let S be a $\widetilde{\gamma}_c^k(G)$-set of cardinality s, and $G_1, G_2, \ldots, G_k$ be the components of $G - S$. Also let $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, k$. Then

$$e \leq \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} + \frac{s(s-1)}{2} + \sum_{i=1}^{k} sn_i.$$

But $\sum n_i = n - s$, and a calculus evaluation shows that the maximum value of $\sum n_i^2$ is $k(\frac{n-s}{k})^2$. So we have

$$(s-n)^2 \geq \frac{2ek - n^2 + nk}{1-k} + n^2.$$

Since $s \leq n$, we obtain $s \leq n - \sqrt{\frac{k(n^2-n-2e)}{k-1}}$. □

The bound of Theorem 2.12 is sharp. For an example if $G = K_n - e$ then $\widetilde{\gamma}_c^2(G) = n - 2 = n - \sqrt{\frac{k(n^2-n-2e)}{k-1}}$ for $k = 2$.

3. Complexity issue for $\widetilde{\gamma}_c^k(G)$

Cyman [1] showed that the following decision problem of the Outer-Connected Dominating Set (OCDS), is an NP-Complete problem.

Outer-Connected Domination Set(OCDS)
Instance: A graph $G = (V, E)$ and a positive integer $m \leq |V(G)|$.
Question: Does G has an outer- connected dominating set of size at most m.

In this section we show that the following decision problem of Outer-k-Connected Dominating Set (OkCDS) is also an NP-complete problem.

Outer-k-Connected Component Domination Set(OkCDS)
Instance: A graph $G = (V, E)$ and a positive integer $m \leq |V(G)|$.
Question: Does G has an outer-k-connected dominating set of size at most m.

To show that (OkCDS) problem is an NP- complete, we reduce this problem to (OCDS) problem by a polynomial time algorithm. First we need to obtain the outer-k-connected component domination number of a disconnected G in terms of the outer-k-connected component domination numbers of its components. For this purpose we define $\widetilde{\gamma}_c^0(G) = |V(G)|$.

Theorem 3.1. *Let G be a disconnected graph with m components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Then*

$$\widetilde{\gamma}_c^k(G) = \min_{k=\sum l_i} \sum_{i=1}^{m} \widetilde{\gamma}_c^{l_i}(G_i)$$

where $l_i \in \{0, 1, 2, \ldots, k\}$.

Proof. Let G be a disconnected graph with m components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Let $S_i^{l_i}$ be a $\widetilde{\gamma}_c^{l_i}(G_i)$-set for $i = 1, 2, \ldots, m$ if G_i has an Ol_iCDS, where $0 \leq l_i \leq k - m + 1$ and $\sum_{i=1}^{m} l_i = k$. It is obvious that $\bigcup_{i=1}^{m} S_i^{l_i}$ is an OkCDS for G. This implies that

$$\widetilde{\gamma}_c^k(G) \leq \min_{k=\sum l_i} \sum_{i=1}^{m} \widetilde{\gamma}_c^{l_i}(G_i).$$

On the other hand let S be an OkCDS for G. Let $S_i = S \cap V(G_i)$ for $i = 1, 2, \ldots, m$. If l_i is the number of components of $G_i \setminus S_i$, then S_i is a Ol_iCDS for G_i. This completes the proof. □

Theorem 3.2. *(OkCDS) for arbitrary graphs is NP-complete.*

Proof. Let H be an arbitrary graph and H_i be a copy of H for $i = 1, 2, \ldots, k-1$. Let G be the graph obtained from $H, H_1, \ldots, H_{k-1}$ by adding a new vertex x_i and joining x_i to every vertex of H_i for $i = 1, 2, \ldots, k-1$. Using Theorem 3.1, we obtain that $\widetilde{\gamma}_c^k(G) = \widetilde{\gamma}_c^1(H) + (k-1)$. Now G has an OkCDS set of size at most m if and only if H has a OCDS set of size at most $m - k + 1$. Since the construction G from H is obtain by a polynomial time algorithm, the proof is complete. □

4. Exact values

In this section we determine the outer-k-connected component domination number in some certain classes of graphs. In particular, with $k = 1$ the $\widetilde{\gamma}_c(G)$ for these graphs ([1]) are consequences of our results.

Proposition 4.1. *For* $n \geq 2$, $\widetilde{\gamma}_c^k(K_n) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$.

Proof. Let $n \geq 2$. If S is an OkCDS in K_n, then $k = 1$, since $K_n \setminus S$ contains exactly one connected component. Now it is obvious that $\widetilde{\gamma}_c^1(K_n) = \gamma(K_n) = 1$ and $\widetilde{\gamma}_c^k(K_n) = 0$ if $k \geq 2$. □

Theorem 4.2. $\widetilde{\gamma}_c^k(P_n) = \begin{cases} 0 & \text{if } n \leq 2k-2 \\ k-1 & \text{if } 2k-1 \leq n \leq 3k-3 \\ k & \text{if } 3k-2 \leq n \leq 3k-1 \\ k+1 & \text{if } n = 3k \\ n-2k & \text{if } n \geq 3k+1. \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, n-1$. Assume that $\widetilde{\gamma}_c^k(P_n) > 0$. Let S be a $\widetilde{\gamma}_c^k(P_n)$-set, and let $G_1, G_2, \dots, G_k$ be the components of $G \setminus S$. Since any vertex in $G \setminus S$ is dominated by some vertex in S, we obtain $n \geq k + (k-1) = 2k-1$. It is obvious that $|S| \geq k-1$, since $G \setminus S$ has k components. We consider the following cases.

Case 1. $n \leq 3k-3$.
Let $S = \{v_{2i} : i = 1, 2, \dots, k-1\}$. Then S is an OkCDS for P_{2k-1}. Consequently, $\widetilde{\gamma}_c^k(P_{2k-1}) = k-1$. So we assume that $n > 2k-1$. Let $n - (2k-1) = i$. We obtain a graph H from P_{2k-1} by subdividing the edge $v_{2j}v_{2j+1}$ and producing two edges $v_{2j}w_j$ and w_jv_{2j+1} for $1 \leq j \leq i$. Then H is a graph isomorphic to P_n. Furthermore, S is an OkCDS for H of cardinality $k-1$. This means that $\widetilde{\gamma}_c^k(P_n) = k-1$.

Case 2. $n \geq 3k-2$.
Any vertex of S dominates at most three vertices of P_n including itself. This implies that $3|S| \geq n$. As a result $|S| \geq k$. If $n = 3k-2$ then $\{v_{3i+2} : 0 \leq i \leq k-2\} \cup \{v_n\}$ is an OkCDS for P_n of cardinality k, and if $n = 3k-1$ then $\{v_{3i+2} : 0 \leq i \leq k-1\}$ is a kCDS for P_n of cardinality k. So for $n \in \{3k-2, 3k-1\}$ we have $\widetilde{\gamma}_c^k(P_n) = k$. Assume next that $n = 3k$. If $|S| = k$, then $S = \{v_{3i+2} : 0 \leq i \leq k-1\}$, since S is a dominating set for P_n. But then $G \setminus S$ has $k+1$ components. This contradiction implies that $|S| \geq k+1$. On the other hand $\{v_{3i+1} : 0 \leq i \leq k-1\} \cup \{v_n\}$ is an OkCDS for P_n of cardinality $k+1$. So $\widetilde{\gamma}_c^k(P_3k) = k+1$. It remains to assume that $n \geq 3k+1$. For $i = 1, 2, \dots, k$, we have $|V(G_i)| \leq 2$. This implies that $|S| \geq n-2k$. On the other hand $\{v_{3i+1} : 0 \leq i \leq k\} \cup \{v_j : j \geq 3k+2\}$ is an OkCDS. Thus $\widetilde{\gamma}_c^k(P_n) = n-2k$. □

The following theorem can be proved in a similar manner as in the proof of Theorem 4.2, and so we omit the proof.

Theorem 4.3. $\widetilde{\gamma}_c^k(C_n) = \begin{cases} 0 & \text{if } n \leq 2k-1 \\ k & \text{if } 2k \leq n \leq 3k-1 \\ n-2k & \text{if } n \geq 3k. \end{cases}$

For a wheel and an integer $k > 1$ the center of the wheel is in any OkCDS. So using Theorem 4.3 we obtain the following.

Corollary 4.4. *Let W_n denots the wheel of order n. Then*

$$\widetilde{\gamma}_c^k(W_n) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } 2 \leq k \text{ and } n \leq 2k-2 \\ k+1 & \text{if } 4 < 2k-1 \leq n \leq 3k-2 \\ n-2k+2 & \text{if } n \geq 3k-1 > 6. \end{cases}$$

Proposition 4.5. *For $m \leq n$,* $\widetilde{\gamma}_c^k(K_{m,n}) = \begin{cases} 0 & \text{if } n < k \\ n & \text{if } k = m = 1 \\ 2 & \text{if } k = 1, m \geq 2 \\ m+n-k & \text{if } n \geq k, k \geq 2. \end{cases}$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the two partite sets of $K_{m,n}$. Assume that $\widetilde{\gamma}_c^k(K_{m,n}) > 0$. So $k \leq n$. If $k = 1$, then $\widetilde{\gamma}_c^1(K_{1,n}) = 1 + n - 1$, and $\widetilde{\gamma}_c^1(K_{m,n}) = \gamma(K_{m,n}) = 2$ when $m \geq 2$. So we assume that $k \geq 2$. Let S be a $\widetilde{\gamma}_c^k(K_{m,n})$-set. Then either $X \subseteq S$ or $Y \subseteq S$, since $G \setminus S$ is disconnected. Without loss of generality assume that $X \subseteq S$. Then $G \setminus S$ is the empty graph which any vertex is a component. But $G \setminus S$ has exactly k connected components. We deduce that $|S| \geq m + (n-k)$. On the other hand $X \cup \{y_1, y_2, \ldots, y_{n-k}\}$ is an OkCDS for $K_{m,n}$ of cardinality $m + n - k$. This completes the proof. □

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