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Construction for antimagic generalized web graphs

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Abstract

An *antimagic labeling* of a graph with q edges is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that all vertex weights are pairwise distinct, where the *vertex weight* is the sum of labels of all edges incident with the vertex.

Let $[n] = \{1, 2, \ldots, n\}$. A *completely separating system* on $[n]$ is a collection $\mathcal{C}$ of subsets of $[n]$ in which, for each pair $a \neq b \in [n]$, there exist $A, B \in \mathcal{C}$ such that $a \in A$, $b \notin A$ and $b \in B$, $a \notin B$.

Recently, a relationship between completely separating systems and labeling of graphs has been shown to exist. Based on this relationship, antimagic labelings of various graphs have been constructed. In this paper, we extend our method to produce more general results for generalized web graphs.

Keywords: antimagic labeling, generalized web graph.

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1. Introduction

All graphs in this paper are finite, simple, undirected and connected. An edge labeling of a graph $G = (V, E)$ is a bijection $l : E \to \{1, 2, \ldots, |E|\}$. The weight of a vertex v, $\mathrm{wt}(v)$, is the sum of the labels of all edges incident with v. Various restrictions on the vertex weights lead to various different types of labelings, for example, the well known magic labelings (see [18]) which have been studied since 1963, supermagic labelings (see [9]). In 1990, Hartsfield and Ringel [8] introduced antimagic labeling of graphs, that is, vertex antimagic edge labeling. An *antimagic labeling* of a graph is a labeling in which the vertex weights are pairwise distinct. A graph is *antimagic* if it has an antimagic labeling.

Hartsfield and Ringel [8] showed that paths, stars, cycles, complete graphs K_m, wheels W_m and bipartite graphs $K_{2,m}$, $m \geq 3$, are antimagic. They also conjectured that every connected graph, except K_2, is antimagic, a conjecture which remains open. Subsequently, several families of graphs have been proved to be antimagic, for example, see [1, 2, 3, 4, 5]. Many other results concerning antimagic graphs are catalogued in the dynamic survey by Gallian [7]. Most recently, new families of antimagic graphs have been discovered by Phanalasy *et al.* [12] and Ryan *et al.* [17].

In this paper, we give an overview of *completely separating systems* (CSSs), define a new family of graphs, the *single apex multi-generalized web graphs*, and show how CSSs may be used to construct antimagic labelings for several families of graphs including this new family of graphs.

Dickson [6] introduced the concept of a completely separating system in 1969. Let $[n] = \{1, 2, \ldots, n\}$. A *completely separating system* on $[n]$ (denoted (n)CSS) is a collection $\mathcal{C}$ of subsets of $[n]$ in which for each pair $a \neq b \in [n]$, there exist $A, B \in \mathcal{C}$ such that $a \in A$, $b \notin A$ and $b \in B$, $a \notin B$.

A *d-element* in a collection of sets is an element which occurs in exactly d sets in the collection. Let k be a positive integer and let $\mathcal{C}$ be an (n)CSS. If $|A| = k$, for all $A \in \mathcal{C}$, then $\mathcal{C}$ is said to be an (n, k)CSS.

Much work has been published on CSSs, for example see [10, 14, 16]. Our initial technique for assigning antimagic labelings using CSSs is based on Roberts' construction for CSSs [15], so it is restated here.

Roberts' construction

Assume that $k \geq 2$, $n \geq \binom{k+1}{2}$ and $k|2n$, and let $R = 2n/k$. An $(R \times k)$-array L is constructed, where each row of L forms a subset of $[n]$ and the R rows of L form an $(n, k)CSS$. Let e_{ij} denote the element of L in row i and column j. Initialize all elements of L to zero. For e from 1 to n, in order, include e in the two positions of L defined by

$$\min_j \min_i \{e_{ij} : e_{ij} = 0\},$$

$$\min_i \min_j \{e_{ij} : e_{ij} = 0\}.$$

That is, e is placed in the first row of L containing a 0, in the first 0-valued place in that row, e is then also placed in the first column of L containing a 0, in the first 0-valued place in that column. Each of the integers 1 to n appears in L in two positions, and the array L is the array of an (n, k)CSS.

The following example illustrates this construction.

Example 1.1. *Using Roberts' construction, we obtain the following array of a* $(6,3)CSS$

1	2	3
1	4	5
2	4	6
3	5	6

The following theorems and lemma will be useful when creating antimagic labelings of the generalized web graphs, so we recall them here. The first two theorems restate Theorem 1 from [12].

Theorem 1.2. *Let* $V = \{v_1, \ldots, v_p\}$ *be a* $(q,k)CSS$ *in which each element is a 2-element. Let* $E = \{\{v_i, v_j\} \mid v_i \cap v_j \neq \emptyset\}$. *Then* $G = (V, E)$ *is a* k*-regular graph and it has an edge labeling* l *given by* $l(\{v_i, v_j\}) = v_i \cap v_j$.

Theorem 1.3. *Let* $G = (V, E)$ *be a* k*-regular graph with* $|V| = p$, $|E| = q$ *and an edge labeling* $l : E \to [q]$. *Then the collection of sets* $\{l(e) \mid v_i \in e\}$ *is a* $(q,k)CSS$ *consisting of 2-elements.*

An edge labeling l will be described using an array L (not necessarily rectangular). Each row of L represents a vertex of G, and each entry of the array is an edge label and must appear in exactly two different rows. Two rows containing the same edge label represent the two vertices incident with the edge with that label. The weight of a row is the sum of its entries; this is the weight of the vertex it represents.

Denote by $G(L)$ the labelled graph determined by the array L.

Theorem 1.4. [12] *Let* L *be the array of a* $(q,k)CSS$ *obtained using Roberts' construction. Then the* k*-regular graph* $G(L)$, *with* q *edges and* $2q/k$ *vertices is antimagic.*

Write G^k for the Cartesian product of k copies of G. The following results will be needed in the next section.

Theorem 1.5. [11] *The Cartesian product graph* $(K_2)^r$, $r \geq 2$, *is antimagic.*

Theorem 1.6. [11] *Let* $G_h = G_h(L_h)$, $h \geq 1$, *where* L_h *is the array of a* $(q_h, k_h)CSS$ *obtained using Roberts' construction. Then*
i) the Cartesian product graph $\prod_h G_h$ *is antimagic;*
ii) the Cartesian product $(K_2)^r \times \prod_h G_h$ $(r \geq 1)$ *is antimagic.*

Lemma 1.7. [13] *Assume that* $H \neq \bigcup_s K_2$. *The disjoint union graph* $H = \bigcup_i H_i$ *is antimagic if each* H_i *is one of* G_j, $(K_2)^r$ $(r \geq 2)$, $\prod_j G_j$ *or* $(K_2)^r \times \prod_j G_j$ $(r \geq 1)$ *and each* G_j *is a graph obtained via Roberts' construction.*

2. Results

For a graph G, the *generalized pyramid graph*, $P(G,1)$ is the graph obtained from G by joining each vertex of G to a vertex called the *apex*.

In the Cartesian product $G \times P_m$ there are m copies of G, say $G_1, \dots, G_m$. We call G_1 and G_m the outer copies of G. The *m-level generalized pyramid graph* or just *generalized pyramid graph*, $P(G,m)$, introduced in [17], is the Cartesian product $G \times P_m$ with a vertex, the apex, joined to each vertex of one of the two outer copies of G.

Proposition 2.1. *Let $G = (V,E)$ be a graph (not necessarily regular) with $|V| > 2$ and with an antimagic labeling. Then $P(G,1)$ is antimagic.*

Proof. The set of vertices of $P(G,1)$ is $V \cup \{a\}$ where a is the apex. Let $V = \{v_1, v_2, \dots, v_p\}$ and $q = |E|$; then $P(G,1)$ has $p+q$ edges, with each new edge containing a. We may assume that in G we have $\mathrm{wt}(v_1) < \mathrm{wt}(v_2) < \cdots < \mathrm{wt}(v_p)$.

Label each new edge $\{a, v_i\}$ with $i+q$. This labeling extends the antimagic labeling of G to an antimagic labeling of $P(G,1)$. $\square$

For G a regular graph there are some constructions for antimagic pyramid graphs for $m > 1$. It can be shown ([13]) that $P(G,m)$ is antimagic for various classes of regular graphs G. For regular graphs G it is possible to generalize the constructions used for $P(G,m)$ to another class of graphs: the web graphs.

The *m-level generalized web graph* (or simply, *generalized web graph*) $WB(G,m)$ is the graph obtained from the m-level generalized pyramid graph $P(G,m)$ by attaching a pendant edge at each vertex of the furthermost copy of G from the apex. When G is a cycle, $WB(G,m)$ is simply called the *web graph*. Example 2.3 shows such a graph.

Denote by T^t the transpose of the array T.

Theorem 2.2. *Let $G_h = G_h(L_h)$, $h \geq 1$, where L_h is the array of a (q_h, k_h)CSS obtained using Roberts' construction. Let $G = G_h$ or $(K_2)^r$ $(r \geq 2)$ or $\prod_h G_h$ or $(K_2)^s \times \prod_h G_h$ $(h, s \geq 1)$. Then the generalized web graph $WB(G,m)$ for $m \geq 1$ is antimagic.*

Proof. Assume that G has p vertices and q edges. An array A representing a labeling of $WB(G,m)$ will be constructed.

Let T_d, $1 \leq d \leq m+1$, be the $(p \times 1)$-array of edges e_i, $1 \leq i \leq p$, where e_i are the edges of $WB(G,m)$ not in any copy of G. Let M_j, $1 \leq j \leq m$, be the array of edge labels of the j-th copy of G.

Case 1. $G = K_p$, $p \geq 3$.

(1) Label the edge e_i, $1 \leq i \leq p$, in the ith row of the array T_d, $1 \leq d \leq m+1$, with $i + (d-1)p$, for $1 \leq d \leq 2$, and $i + (d-1)p + (d-2)q$, for $2 < d \leq m+1$;

(2) Relabel the edge labels in the array M_j, $1 \le j \le m$, by adding $(j+1)p + (j-1)q$ to each of the original edge labels;

(3) Define A to be

$$\begin{array}{ccc} & & T_1 \\ & & T_2^t \\ T_1 & T_2 & M_1 \end{array} \quad \text{when } m = 1 \text{ and } \quad \begin{array}{ccc} & & T_1 \\ & & T_2^t \\ T_1 & M_1 & T_3 \\ T_2 & M_2 & T_4 \\ \vdots & \vdots & \vdots \\ T_{m-1} & M_{m-1} & T_{m+1} \\ T_m & T_{m+1} & M_m \end{array} \quad \text{when } m \ge 2.$$

Case 2. $G \ne K_p$.

(1) Label the edge e_i, $1 \le i \le p$, in row i of the array T_d, $1 \le d \le m+1$, with $i+(d-1)(p+q)$;

(2) Relabel the edge labels in the array M_j, $1 \le j \le m$, by adding $jp + (j-1)q$ to each of the original edge labels;

(3) Define A to be

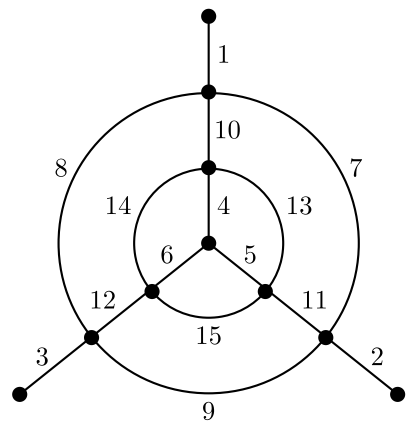
$$\begin{array}{ccc} & & T_1 \\ T_1 & M_1 & T_2 \\ T_2 & M_2 & T_3 \\ \vdots & \vdots & \vdots \\ T_{m-1} & M_{m-1} & T_m \\ T_m & M_m & T_{m+1} \\ & & T_{m+1}^t \end{array} \tag{1}$$

In each case it is clear that the weight of each row is less than the weight of the row below and so the labeling of $WB(G, m)$ is antimagic. □

The following example illustrates the construction of an antimagic labeling of the web graph $WB(K_3, 2)$.

Example 2.3. *Using the construction in the proof of Theorem 2.2, we have the array A of edge labels of the web graph $WB(K_3, 2)$ and the corresponding graph with an antimagic labeling:*

			1
			2
			3
	4	5	6
1	7	8	10
2	7	9	11
3	8	9	12
4	10	13	14
5	11	13	15
6	12	14	15



The only possible edge labeling of K_2 can be presented by the (2×1)-array, M_0, with both entries 1. Interestingly, the same construction as given in the proof of Case 1 of Theorem 2.2 also works when the array M_j is replaced by M_{0j} (M_{0j} is the array of edge labels of the j-th copy of K_2), although K_2 itself is not antimagic. Then we have

Theorem 2.4. *The generalized web graph $WB(K_2, m)$, $m \geq 1$, is antimagic.*

Surprisingly, when the array M_j, $1 \leq j \leq m$, is removed from the construction of Case 1 of Theorem 2.2, we obtain an alternative proof of the following theorem. The reason this works is that K_1 has no edges, and hence the array of edge labels is empty. The path P_n, $n \geq 3$, has been proved to be antimagic in [8].

Theorem 2.5. *The generalized web graph $WB(K_1, m) = P_{m+2}$, $m \geq 1$, is antimagic.*

3. Further Results

In this section we introduce the single apex multi-generalized web graph and prove that for various underlying graphs it is antimagic.

We begin with a graph H which is the union of connected components $G_1, \ldots, G_r$. For each connected component G_i construct $WB(G_i, m)$, then identify the apexes of these graphs. The identified apexes give a single apex in the new graph, so the new graph is called the *single apex multi-generalized web graph* $MWB(H, m)$. The single apex multi-generalized web graph for $H = C_3 \cup C_4$ and $m = 2$ is shown in Figure 1. When H is a connected graph $MWB(H, m) = WB(H, m)$.

Theorem 3.1. *Let $H = \bigcup_s H_s$ be any of the antimagic k-regular disconnected graphs which appear in Lemma 1.7. Then the single apex multi-generalized web graph $MWB(H, m)$, $m \geq 1$, is antimagic.*

Proof. By Lemma 1.7 there exists an array M with k columns representing H, together with a labeling, with the weight of each row of M greater than the weight of the row above.

Let M_j, $1 \le j \le m$, be the array of edge labels of the j-th copy of H. The construction presented in Case 2 of the proof of Theorem 2.2 applies here. □

Example 3.2. *The arrays used in the construction of the single apex multi-generalized web graph $MWB(C_3 \cup C_4, 2)$ are*

$$T_1 = \begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array},\ T_2 = \begin{array}{c} 15\\16\\17\\18\\19\\20\\21 \end{array},\ T_3 = \begin{array}{c} 29\\30\\31\\32\\33\\34\\35 \end{array},\ M_1 = \begin{array}{cc} 8 & 9\\ 8 & 10\\ 9 & 10\\ 11 & 12\\ 11 & 13\\ 12 & 14\\ 13 & 14 \end{array},\ \textit{and } M_2 = \begin{array}{cc} 22 & 23\\ 22 & 24\\ 23 & 24\\ 25 & 26\\ 25 & 27\\ 26 & 28\\ 27 & 28 \end{array}.$$

From these arrays one builds the array (1) as shown in the proof of Theorem 2.2, which represents the labeled graph shown in Figure 1.

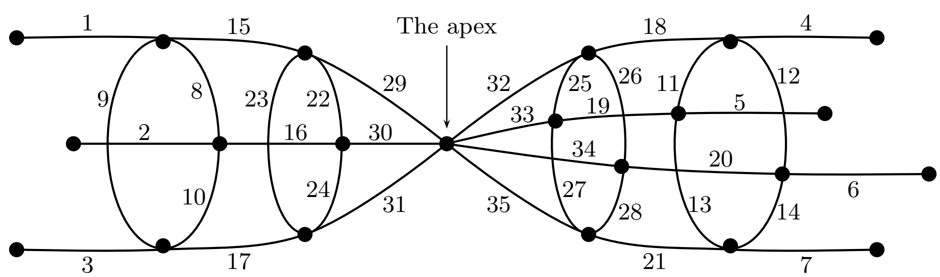


Figure 1: The graph $MWB(C_3 \cup C_4, 2)$ with an antimagic labeling

Write kG for the union of k copies of G. The array $M = (1, 1, 2, 2, \ldots, a, a)^t$ represents the graph aK_2. The labeling that this represents is not antimagic; aK_2 is not an antimagic graph. However, the construction given in Case 2 of the proof of Theorem 2.2 applies using M. Then we have

Theorem 3.3. *The single apex multi-generalized web graph $MWB(aK_2, m)$, $m \ge 1$, is antimagic.*

Note that $MWB(aK_1, m)$ is a regular subdivision of a star. The construction given in Case 2 of the proof of Theorem 2.2 also works when each M_j is removed from the construction. This proves

Theorem 3.4. *The regular subdivision of a star, $MWB(aK_1, m)$ $(m \ge 1)$, is antimagic.*

When $a = 2$, $MWB(2K_1, m) = P_{2m+3}$; it is a special case of Theorem 2.5.

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