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Further Results on Harmonious Colorings of Digraphs*

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Abstract

Let D be a directed graph with n vertices and m edges. A function $f : V(D) \to \{1, 2, 3, ..., t\}$, where $t \leq n$ is said to be a *harmonious coloring* of D if for any two edges xy and uv of D, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If no pair (i, i) is assigned, then f is said to be a *proper harmonious coloring* of D. The minimum t for which D admits a proper harmonious coloring is called the *proper harmonious coloring number* of D. We investigate the proper harmonious coloring number of graphs such as alternating paths and alternating cycles.

Keywords: harmonious coloring, proper harmonious coloring number, digraphs

2010 Mathematics Subject Classification: 05C78, 05C15, 05C20.

1. Introduction

In this paper, we consider only finite simple graphs. For all notations in graph theory we follow Harary [3], West [6] and Chartrand [1]. Coloring the vertices and edges of a graph which is required to obey certain conditions, have often been motivated by their utility to various applied fields and their mathematical interest. Various coloring problems such as the vertex coloring and edge coloring problem have been studied in the literature [3].

Definition 1.1. *A coloring of a graph G is a function $c : V(G) \to X$ for some set of colors X such that $c(u) \neq c(v)$ for each edge $uv \in E(G)$.*

The coloring defined above is the vertex coloring where we color the vertices of a graph such that no two adjacent vertices are colored with the same color. Similarly the edge coloring problem can be defined in such a way that no two adjacent edges are colored the same color. Hopcroft and Krishnamoorthy [4] introduced a type of edge coloring called harmonious coloring.

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Definition 1.2. *A* harmonious coloring *[4] of a graph G is an assignment of colors to the vertices of G and the color of an edge is defined to be the unordered pair of colors to its end vertices such that all edge colors are distinct. The* harmonious coloring number *is the least number of colors needed in such a coloring.*

Considerable body of literature has grown around the subject Harmonious Coloring. The list of articles published on the subject can be found in [2].

The following is an extension of harmonious coloring to directed graphs given by Hegde and Castelino [5].

Definition 1.3. *Let D be a directed graph with n vertices and m edges. A function $f : V(D) \to \{1, 2, ..., t\}$, where $t \leq n$ is said to be a harmonious coloring of D if for any two edges xy and uv of D , the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If no pair (i, i) is assigned, then f is called a proper harmonious coloring of D. The minimum t for which D admits a proper harmonious coloring is called the proper harmonious coloring number of D and is denoted by $\overrightarrow{\chi_h}(D)$.*

The following results have been proved by Hegde and Castelino [5].

Proposition 1.4. *The proper harmonious coloring number of a symmetric digraph is same as the proper harmonious coloring number of its underlying graph.*

Proposition 1.5. *Let D be a directed graph with n vertices. Then $\Delta + 1 \leq \overrightarrow{\chi_h}(D) \leq n$, where Δ is the maximum indegree or outdegree of any vertex v of D.*

The harmonious coloring number of a star $\overrightarrow{S}_n$ with n vertices is $\Delta + 1$ and the harmonious coloring number of a complete symmetric digraph $\overleftrightarrow{K}_n$ with n vertices is n.

Proposition 1.6. *For any digraph D, $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of edges of D.*

Let $k = \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$ be a parameter corresponding to a graph G with n vertices and m edges, where $\lceil x \rceil$ denotes the least integer which is greater than or equal to real x. Unless mentioned otherwise, we mean $k = \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$ throughout this paper.

Proposition 1.7. *Let $\overrightarrow{P}_n$ be a unipath with n vertices. Then $\overrightarrow{\chi_h}(\overrightarrow{P}_n) = k$.*

Proposition 1.8. *Let $\overrightarrow{C}_n$ be a unicycle with n vertices, then,*

$$\overrightarrow{\chi_h}(\overrightarrow{C}_n) = \begin{cases} k+1 & for \quad n = k(k-1)-1, \\ k & for \quad n = (k-1)(k-2)+1, ..., k(k-1)-2, k(k-1). \end{cases}$$

2. Proper Harmonious Colorings of Some Classes of Digraphs

In this section we present some results on the proper harmonious coloring number of some classes of digraphs.

In general, the proper harmonious coloring problem has been viewed as an Eulerian path decomposition of graphs. Whenever such a decomposition possible, it is possible to find the proper harmonious coloring number of graphs [4].

Definition 2.1. *An alternating Eulerian trail of a digraph D is an open trail of D including all the edges and vertices of D with every adjacent edge having opposite direction.*

Lemma 2.2. *If G is a connected (undirected) non-bipartite graph in which every vertex has even degree, then the symmetric digraph $D(G)$ obtained from G has an alternating closed Eulerian trail.*

Proof. Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Since G is non-bipartite, it has a cycle of odd length. Also, since all the vertices of G are of even degree, G is Eulerian. That is G has a closed Eulerian trail say, T. Now, consider the symmetric digraph $D(G)$ obtained from G by replacing each undirected edge by a pair of edges with opposite orientations. Since G has a closed Eulerian trail T, we get a closed Eulerian trail in $D(G)$ in which the adjacent edges have opposite direction as follows:

Suppose G has m edges. Then there exists two cases.

Case (i) Let m be odd.

We obtain the required alternating closed Eulerian trail in $D(G)$ by traversing T twice in the same direction but using the edges of alternating direction.

Case (ii) Let m be even.

We observe that T must visit some vertex v twice with an odd number of edges between the visits (otherwise G would be bipartite). Hence we get T as $v_0 = v, v_1, \cdots, v_i = v, v_{i+1}, \cdots, v_m = v$, where i is odd. Then the required alternating closed Eulerian trail is $v_0 = v \rightarrow v_1 \leftarrow \cdots \rightarrow v_i = v \leftarrow v_1 \rightarrow \cdots \leftarrow v_i = v \rightarrow v_{i+1} \leftarrow \cdots \rightarrow v_m = v \leftarrow v_{i+1} \rightarrow \cdots \leftarrow v_m = v$. □

As a consequence of the above lemma, we have the following lemma.

Lemma 2.3. *The alternating cycle on n vertices can be colored with k colors if there is a connected (undirected) non-bipartite graph G, with every vertex having even degree and with k vertices and $\frac{n}{2}$ edges.*

Definition 2.4. *An alternating path $\overrightarrow{AP_n}$ with n vertices is an oriented path in which any two consecutive edges have opposite directions.*

Theorem 2.5. *Let* $\overrightarrow{AP_n}$ *be an alternating path. Then*

$$\overrightarrow{\chi_h}(\overrightarrow{AP_n}) = \begin{cases} k+1 & \text{if } k \text{ is even and } k^2-2k+3 \leq n \leq k^2-k+1 \\ k & \text{otherwise.} \end{cases}$$

Proof. Since $\overrightarrow{AP_n}$ is an alternating path with n vertices and $n-1$ edges, by Proposition 1.6, we get $\overrightarrow{\chi_h}(\overrightarrow{AP_n}) \geq k$. When $\overrightarrow{\chi_h}(\overrightarrow{AP_n}) = k$, it follows that $k^2-3k+4 \leq n \leq k^2-k+1$. Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Then the proper harmonious coloring number of $\overrightarrow{AP_n}$ is equivalent in finding an alternating Eulerian trail by traversing in the same direction but using edges of opposite direction in $D(G)$. Now, consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd and let $G = K_k$. Then G contains $\frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 2.2, we obtain the required alternating closed Eulerian trail in $D(G)$.

Case (ii) Let k be even.

Case (a) Let $k=4$ and let v_1, v_2, v_3 and v_4 be the vertices of K_4. Then we can find an alternating Eulerian trail in $\overleftrightarrow{K_4}$ as follows: $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_1 \leftarrow v_2 \rightarrow v_4 \leftarrow v_1 \rightarrow v_3 \leftarrow v_4 \rightarrow v_1$.

Case (b) Let $k \geq 6$ and let $G = K_k \setminus M$, where M is the matching of size $k/2$ (k should be at least 6 so that G is not bipartite). Then G will have k vertices and $\frac{k^2-2k}{2}$ edges. Also, all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. As m will be even for any value of k, by Lemma 2.2, we can find an alternating closed Eulerian trail in $D(G)$ and the length of this alternating closed Eulerian trail is $2(\binom{k}{2} - k/2) = k^2-2k$. Regarding this as an open alternating trail, we can clearly add one further edge to one end of it in $D(G)$ using one edge of the matching in one direction. Hence we will get an alternating Eulerian trail of length k^2-2k+1.

We know that since k colors are used to color the vertices of $\overrightarrow{AP_n}$ of length $n-1$, there will be $k(k-1)$ ordered pairs of colors. In $\overrightarrow{AP_n}$, at each vertex there will be either two incoming edges or two outgoing edges except for the first and the last vertex. Hence it requires even number of ordered pairs at each vertex. There will be $k-1$ ordered pairs associated with each color. When k is even, $k-1$ will be odd and hence we cannot use $k-1$ ordered pairs of one particular color. That is only k^2-2k+1 ordered pairs of colors will be used when k is even. Hence when k is even, for an alternating path with more than k^2-2k+2 vertices, we require $k+1$ colors. Hence the proof. □

Figures 1, 2 and 3 are the illustrative examples for the above result.

Definition 2.6. *An alternating cycle* $\overrightarrow{AC_n}$ *with* n *vertices is an oriented cycle in which all adjacent edges have opposite directions.*

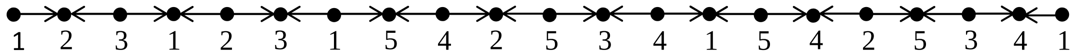


Figure 1: Proper harmonious coloring of $\overrightarrow{AP}_{21}$.

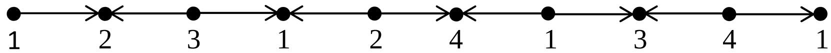


Figure 2: Proper harmonious coloring of $\overrightarrow{AP}_{10}$.

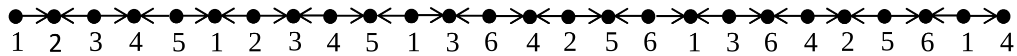


Figure 3: Proper harmonious coloring of $\overrightarrow{AP}_{26}$.

Let $\overleftrightarrow{K_k}$ be a complete symmetric digraph. Then we have the following results:

Lemma 2.7. *Let the alternating cycle $\overrightarrow{AC_n}$ be a subgraph of $\overleftrightarrow{K_k}$ of length n. Then every vertex of $\overrightarrow{AC_n}$ in $\overleftrightarrow{K_k}$ has even indegree and outdegree.*

Proof. Let $\overrightarrow{AC_n}$ be any alternating cycle of length n in $\overleftrightarrow{K_k}$. By definition, an alternating cycle is a cycle in which any two consecutive edges have opposite directions. Hence any vertex of $\overrightarrow{AC_n}$ in $\overleftrightarrow{K_k}$ should have either two incoming edges or two outgoing edges. Hence every vertex of $\overrightarrow{AC_n}$ will have even indegree and outdegree. Hence the proof. □

Lemma 2.8. *Let the alternating cycle $\overrightarrow{AC_n}$ be a subgraph of $\overleftrightarrow{K_k}$. When k is odd, for $n = k(k-1) - 2$, $\overrightarrow{AC_n}$ cannot be colored with k colors and hence requires $k+1$ colors.*

Proof. Let us assume that $k(k-1)-2$ vertices can be colored with k colors. Then in $\overleftrightarrow{K_k}$, the possible degree sequence of the outgoing edges will be $(k-1, k-1, \ldots, (k-1) times, k-3)$. Then corresponding to this degree sequence of outgoing edges, we get the degree sequence of the incoming edges as $(k-1, k-1, \ldots, (k-2) times, k-2, k-2)$. Hence there exists at least two vertices having odd indegrees, a contradiction by Lemma 2.7. Thus, at least $k+1$ colors are required. □

Lemma 2.9. *Let $\overrightarrow{AC_n}$ be an alternating cycle with n vertices. Then when k is even, $k(k-2) + 2 \leq n \leq k(k-1)$ vertices cannot be colored with k colors and hence requires $k+1$ colors.*

Proof. The total indegree and the total outdegree of those vertices of the alternating cycle having any particular color must both be even (by Lemma 2.7) and so if k is even, they cannot exceed $k-2$ (as there are only $k-1$ ordered pairs of one particular color). It follows that there can be at most $k(k-2)$ edges in the alternating cycle with k colors.

Hence for $\overrightarrow{AC_n}$ with n vertices , where $k(k-2)+2 \leq n \leq k(k-1)$, we require one more additional color to color the vertices. □

Theorem 2.10. *Let $\overrightarrow{AC_n}$ be an alternating cycle with n vertices, where n is even. Then*

$$\overrightarrow{\chi_h}(\overrightarrow{AC_n}) = \begin{cases} k & \text{for } k = \text{odd and } n = (k-1)(k-2)+2, \ldots, k(k-1)-4, k(k-1) \\ k+1 & \text{for } k = \text{odd and } n = k(k-1)-2 \\ k & \text{for } k = \text{even and } n = (k-1)(k-2)+2, \ldots, k(k-2) \\ k+1 & \text{for } k = \text{even and } n = k(k-2)+2, k(k-2)+4, \ldots, k(k-1). \end{cases}$$

Proof. Since an alternating cycle $\overrightarrow{AC_n}$ contains n edges, by Proposition 1.6, we get $\overrightarrow{\chi_h}(\overrightarrow{AC_n}) \geq k$. When $\overrightarrow{\chi_h}(\overrightarrow{AC_n}) = k$, it follows that $(k-1)(k-2)+2 \leq n \leq k(k-1)$.

Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Now, consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd.

Case (a) Let $n = k(k-1)$, $k \geq 3$ and let $G = K_k$. Then G contains $\frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 2.3, we can find an alternating cycle of length n.

Case (b) Let $n = (k-1)(k-2)+2, \cdots, k(k-1)-6$, $k \geq 5$ and let $G = K_k \setminus C_t$, where C_t is a cycle with t vertices, $t = 3, 4, \cdots, k-2$. Then G has $\frac{k(k-1)-2t}{2}$ edges , where $t = 3, 4, \cdots, k-2$. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3, we obtain the required alternating cycle of length n.

Case (c) Let $n = k(k-1)-4$, $k \geq 5$. Let $v_1, v_2, v_3, \cdots, v_k$ be the vertices of K_k. Let $G = K_k \setminus C_4$, where C_4 is a cycle v_1, v_2, v_3, v_4, v_1 of length 4. Then G has $\frac{k(k-1)}{2} - 4$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.2, we obtain the alternating closed Eulerian trail of length $k(k-1)-8$ in $D(G)$. Now, suppose the alternating closed Eulerian trail contains $\cdots \rightarrow v_1 \leftarrow \cdots$, and add in the edges $v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4 \rightarrow v_1$. Then we obtain the required alternating closed Eulerian trail of length $k(k-1)-4$.

Case (ii) Let k be even and let $G_1 = K_k \setminus M_1$, where M_1 is the matching of size $k/2$ (k should be at least 6 so that G_1 is not bipartite).

Case (a) When $n = k(k-2)$ and $k = 4$, we can color the vertices of $\overrightarrow{AC_8}$ as given below:

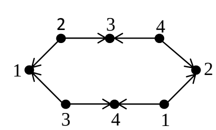
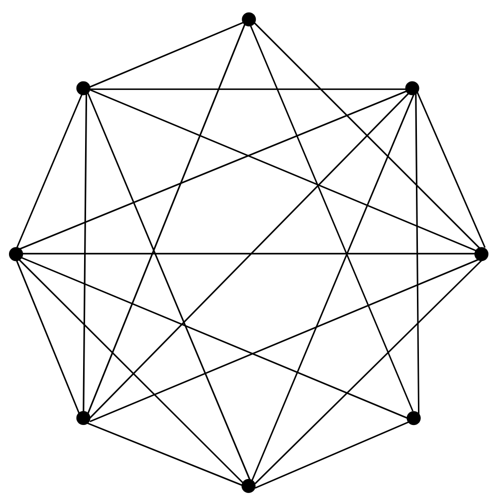


Figure 4: Proper harmonious coloring of $\overrightarrow{AC_8}$.

Case (b) Let $n = k(k-2)$, $k \geq 6$ and let $G = G_1$. Then G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Also G contains $\frac{k^2-2k}{2}$ edges. Then by Lemma 2.3, we obtain the required alternating cycle of length $k^2 - 2k$.

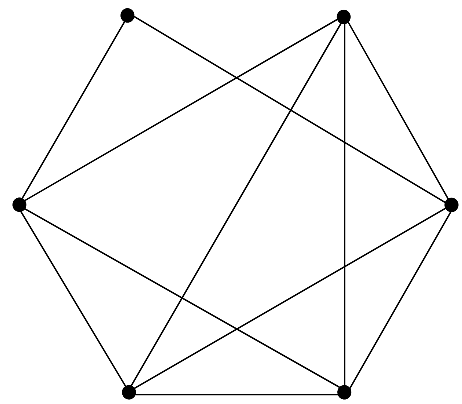
Case (c) Let $n = (k-1)(k-2)+2, (k-1)(k-2)+4, \cdots, k(k-1)-6$, $k \geq 10$ and let $G = G_1 \setminus C_t$, where C_t is a cycle with t vertices, $t = 3, 4, \cdots, (\frac{k}{2}-2)$. Then G has $\frac{k^2-2(k-t)}{2}$ edges for $t = 3, 4, \cdots, (\frac{k}{2}-2)$. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3, we obtain the required alternating cycle of length n.

Case (d) Let $n = k(k-2)-4$, $k \geq 8$ and let $G_2 = K_k \setminus M_2$, where M_2 is the matching of size $\frac{k}{2} - 2$ (k should be at least 6 so that G_2 is not bipartite). Then $G = G_2 \setminus 2P_3$, where P_3 is a path of length 2 and the end vertices of both the paths are the vertices which are not the adjacent vertices of the edges of the matching. Also, both the paths are distinct and passes through the vertex which is incident with the edge of the matching. The following sketch illustrates G when $k = 8$.



Then G has $\frac{k(k-1)-k-4}{2}$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3, we obtain the required alternating cycle of length n.

Case(e) Let $n = k(k-2)-2$, $k \geq 6$ and let $G_3 = K_k \setminus M_3$, where M_3 is the matching of size $\frac{k}{2} - 1$ (k should be at least 6 so that G_3 is not bipartite). Then $G = G_3 \setminus P_3$, where P_3 is a path of length 2 and the end vertices of P_3 are the vertices which are not the adjacent vertices of the edges of the matching. Also, it passes through the vertex which is incident with the edge of the matching. Consider the sketch below as an example of G for the case when $k = 6$.



Then G has $\frac{k(k-1)-k-2}{2}$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3, we obtain the required alternating alternating cycle of length n.

We can conclude the result using Lemma 2.8 and Lemma 2.9. □

Figures 5 and 6 are the illustrative examples for the above result.

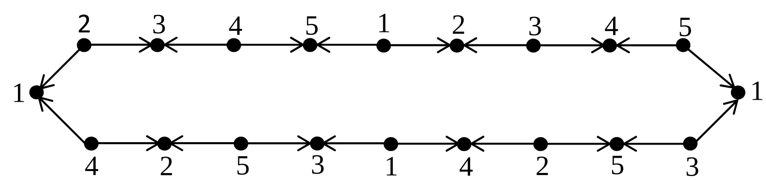


Figure 5: Proper harmonious coloring of $\overrightarrow{AC}_{20}$.

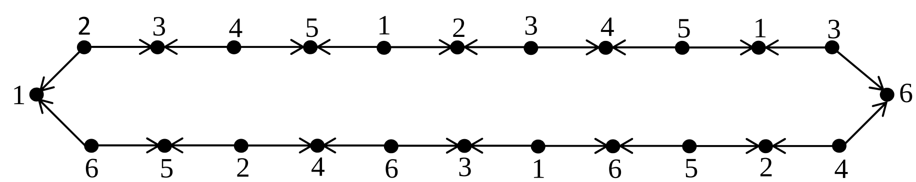


Figure 6: Proper harmonious coloring of $\overrightarrow{AC}_{24}$.

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