AKCE Int. J. Graphs Comb., 8, No. 2 (2011), pp. 161-168

The (j,k)-domatic number of a graph

S.M. Sheikholeslami* and L. Volkmann†

*Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran.
e-mail: *s.m.sheikholeslami@azaruniv.edu*

†Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany.
volkm@math2.rwth-aachen.de

Communicated by: S. Arumugam
Received 03 September 2010; accepted 26 September 2011

Abstract

Let $k \geq j \geq 1$ be two integers, and let G be a simple graph such that $j(\delta(G)+1) \geq k$, where $\delta(G)$ is the minimum degree of G. A (j,k)-*dominating function* of a graph G is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,j\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u\in N[v]} f(u) \geq k$ is fulfilled, where $N[v]$ is the closed neighborhood of v. A set $\{f_1, f_2, \ldots, f_d\}$ of (j,k)-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq j$ for each $v \in V(G)$, is called a (j,k)-*dominating family* (of functions) on G. The maximum number of functions in a (j,k)-dominating family on G is the (j,k)-*domatic number* of G, denoted by $d_{(j,k)}(G)$. Note that $d_{(1,1)}(G)$ is the classical domatic number $d(G)$. In this paper we initiate the study of the (j,k)-domatic number in graphs and we present some bounds for $d_{(j,k)}(G)$. Many of the known bounds of $d(G)$ are immediate consequences of our results.

Keywords: (j,k)-dominating function, (j,k)-domination number, (j,k)-domatic number.

2010 Mathematics Subject Classification: 05C69

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Consult [5, 10] for the notation and terminology which are not defined here.

Let $k \geq j \geq 1$ be two integers, and let G be a simple graph such that $j(\delta(G)+1) \geq k$. A (j,k)-*dominating function* ((j,k)DF) of a graph G is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,j\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u\in N[v]} f(u) \geq k$ is fulfilled. The *weight* of a (j,k)DF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The (j,k)-*domination number* of a graph G, denoted by $\gamma_{(j,k)}(G)$, is the minimum weight of a (j,k)DF of G. By assumption,

$$\gamma_{(j,k)}(G) \geq k. \tag{1}$$

As the assumption $j(\delta(G)+1) \geq k$ is clearly necessary, we always assume that when we discuss $\gamma_{(j,k)}(G)$, all graphs involved satisfy $j(\delta(G)+1) \geq k$. A $\gamma_{(j,k)}(G)$-*function* is a (j,k)-dominating function of G with weight $\gamma_{(j,k)}(G)$. Note that $\gamma_{(1,1)}(G)$ is the classical domination number $\gamma(G)$. The (j,k)-domination number was introduced by Rubalcaba and Slater [6] and has been studied by several authors (see for example [7]). If $j = 1$, then $(1,k)$-domination is k-tuple domination ([3]), thus for any graph G, $\gamma_{(1,k)}(G) = \gamma_{\times k}(G)$. If $j = k$, then (k,k)-domination is $\{k\}$-domination ([1, 9]), thus for any graph G, $\gamma_{(k,k)}(G) = \gamma_{\{k\}}(G)$. It is easy to see that

$$\gamma_{(j,k)}(G) \leq j\gamma(G) \tag{2}$$

for any $k \geq j \geq 1$ and any graph G.

Let $k \geq j \geq 1$ be two integers, and let G be a simple graph in which $j(\delta(G)+1) \geq k$. A set $\{f_1, f_2, \ldots, f_d\}$ of (j,k)-dominating functions of G with the property that $\sum_{i=1}^{d} f_i(v) \leq j$ for each $v \in V(G)$, is called a (j,k)-*dominating family* (of functions) on G. The maximum number of functions in a (j,k)-dominating family ((j,k)D family) on G is the (j,k)-*domatic number* of G, denoted by $d_{(j,k)}(G)$. By assumption, the (j,k)-domatic number is well-defined and

$$d_{(j,k)}(G) = 0 \tag{3}$$

if $j(\delta(G)+1) < k$ and

$$d_{(j,k)}(G) \geq 1 \tag{4}$$

for all graphs G with $j(\delta(G)+1) \geq k$, since the set consisting of the function $f : V(G) \to \{0,1,2,\ldots,j\}$ defined by $f(v) = j$ for each $v \in V(G)$, forms a (j,k)D family on G. Note that $d_{(1,1)}(G)$ is the classical domatic number $d(G)$. If $j = 1$, then $(1,k)$-domatic is k-tuple domatic ([3]), thus for any graph G, $d_{(1,k)}(G) = d_{\times k}(G)$.

Our purpose in this paper is to initiate the study of the (j,k)-domatic number in graphs. We first study basic properties and bounds for the (j,k)-domatic number of a graph. In addition, we determine the (j,k)-domatic number of some classes of graphs.

We finish this section with determining the (j,k)-domatic number of complete graphs.

Proposition 1.1. *For any three positive integers n, k, j with $k \geq j \geq 1$ and $nj \geq k$, $\gamma_{(j,k)}(K_n) = k$ and $d_{(j,k)}(K_n) = \lfloor \frac{jn}{k} \rfloor$.*

Proof. Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ and let $k=rj+s$, where r,s are nonnegative integers with $0\le s<j$. Define $f:V(G)\to\{0,1,2,\ldots,j\}$ by

$$f(v_i)=j \text{ for } 1\le i\le r\ ,\ f(v_i)=1 \text{ for } r+1\le i\le r+s \text{ when } s>0 \text{ and } f(v_i)=0 \text{ otherwise.}$$

Obviously, f is a (j,k)-dominating function of G with $w(f)=k$. Hence, $\gamma_{(j,k)}(K_n)=k$ by (1).

To prove $d_{(j,k)}(K_n)=\lfloor\frac{jn}{k}\rfloor$, we use induction on n. For $n=1,2$, the equality is clearly true. Let $n\ge 3$ and the equality is valid for all complete graphs on at most $n-1$ vertices. Let $k\ge j$ be two positive integers in which $jn\ge k$. Suppose that $n=kd_{\times k}+s$, where $0\le s<k$ and $js=kt+s'$, where t,s' are positive integers and $0\le s'<k$. Then $nj=k(jd_{\times k}+t)+s'$ and $\lfloor\frac{jn}{k}\rfloor=jd_{\times k}+t$.

Now consider a partition of $V(K_n)$ into $d_{\times k}+1$ sets $V_1,V_2,\ldots,V_{d_{\times k}},S$ such that V_i's are k-tuple dominating sets of size k and $|S|=s$. Now we associate j, (j,k)-dominating functions to each V_i and t, (j,k)-dominating functions to S. Assume that $k=jr+r'$, where r,r' are positive integers and $0\le r'<j$. Let $V_i^1,V_i^2,\ldots,V_i^j,S_i$ be a partition of V_i, where $|V_i^1|=|V_i^2|=\ldots=|V_i^j|=r$ and $|S_i|=r'$. For $1\le i\le d_{\times k}$ and $1\le m\le j$, define $f_i^m:V(G)\to\{0,1,2,\ldots,j\}$ by

$$f_i^m(x)=j \text{ if } x\in V_i^m\ ,\ f_i^m(x)=1 \text{ if } x\in S_i \text{ and } f_i^m(x)=0 \text{ otherwise.}$$

On the other hand, by the induction hypothesis $d_{(j,k)}(K_s)=\lfloor\frac{js}{k}\rfloor=t$ when $js\ge k$ and $d_{(j,k)}(K_s)=0$ if $js<k$. If $js\ge k$, let $\{g_1,g_2,\ldots,g_t\}$ be a (j,k)D family on the complete graph with vertex set S. For $1\le l\le t$, define $g_l^*:V(G)\to\{0,1,2,\ldots,j\}$ by

$$g_l^*(x)=g_l(x) \text{ if } x\in S \text{ and } g_l^*(x)=0 \text{ if } x\in V(G)\setminus S.$$

It is easy see that $\{f_i^m\mid 1\le i\le d_{\times k},1\le m\le j\}$ when $js<k$ and $\{f_i^m,g_l^*\mid 1\le i\le d_{\times k},1\le m\le j, \text{ and } 1\le l\le t\}$ when $js<k$, is a (j,k)D family on K_n and the proof is complete. □

Corollary 1.2. [4] *For any two positive integers $n\ge k$, $\gamma_{\times k}(K_n)=k$ and $d_{\times k}(K_n)=\lfloor\frac{n}{k}\rfloor$.*

2. Properties of the (j,k)-domatic number

In this section we mainly present basic properties of $d_{(j,k)}(G)$ and bounds on the (j,k)-domatic number of a graph.

Theorem 2.1. *If G is a graph of order n, then*

$$\gamma_{(j,k)}(G)\cdot d_{(j,k)}(G)\le jn.$$

Moreover, if $\gamma_{(j,k)}(G)\cdot d_{(j,k)}(G)=jn$, then for each $(j,k)D$ family $\{f_1,f_2,\ldots,f_d\}$ on G with $d=d_{(j,k)}(G)$, each function f_i is a $\gamma_{(j,k)}(G)$-function and $\sum_{i=1}^{d}f_i(v)=j$ for all $v\in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a (j,k)D family on G such that $d = d_{(j,k)}(G)$. Then

$$\begin{aligned} d \cdot \gamma_{(j,k)}(G) &= \sum_{i=1}^{d} \gamma_{(j,k)}(G) \le \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V} j = jn. \end{aligned}$$

If $\gamma_{(j,k)}(G) \cdot d_{(j,k)}(G) = jn$, then the two inequalities occurring in the proof become equalities. Hence for the (j,k)D family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{(j,k)}(G)$. Thus each function f_i is a $\gamma_{(j,k)}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = j$ for all $v \in V$. □

The case $(1,1)$ in Theorem 2.1 leads to the well-known inequality $\gamma(G) \cdot d(G) \le n$, given by Cockayne and Hedetniemi [2] in 1977, and the case (k,k) in Theorem 2.1 leads to the inequality $\gamma_{\{k\}}(G) \cdot d_{\{k\}}(G) \le nk$, given by Meierling, Sheikholeslami and Volkmann [8].

Theorem 2.2. *If $k \ge j \ge 1$ are two positive integers, and G is a graph of order n, then*

$$d_{(j,k)}(G) \le \frac{jn}{k},$$

with equality if and only if k is a divisor of jn and G is isomorphic to the complete graph K_n.

Proof. If $j(\delta(G)+1) < k$ then by (3), $d_{(j,k)}(G) = 0 < \frac{jn}{k}$. Let $j(\delta(G)+1) \ge k$. Since $\gamma_{(j,k)}(G) \ge k$, it follows from Theorem 2.1 that

$$d_{(j,k)}(G) \le \frac{jn}{\gamma_{(j,k)}(G)} \le \frac{jn}{k},$$

and the desired inequality is proved. If k is a divisor of jn and G is isomorphic to the complete graph K_n, then Proposition 1.1 implies that $d_{(j,k)}(G) = \frac{jn}{k}$.

Conversely, let $d_{(j,k)}(G) = \frac{jn}{k}$. As $d_{(j,k)}(G)$ is an integer, we deduce that k is a divisor of jn. In addition, Theorem 2.1 leads to $\gamma_{(j,k)}(G) = k$. Let now $\{f_1, f_2, \ldots, f_d\}$ be a (j,k)D family on G with $d = d_{(j,k)}(G)$. Theorem 2.1 implies that $\sum_{x \in V(G)} f_i(x) = \gamma_{(j,k)}(G) = k$ for each $i \in \{1, 2, \ldots, d\}$ and $\sum_{i=1}^{d} f_i(x) = j$ for all $x \in V(G)$. Let $v \in V(G)$ be an arbitrary vertex of G. The identity $\sum_{i=1}^{d} f_i(v) = j$ yields to $f_s(v) > 0$ for at least one index s. If $u \in V(G) \setminus \{v\}$ is an arbitrary vertex, then we obtain

$$k = \sum_{x \in V(G)} f_s(x) \ge \sum_{x \in N[u]} f_s(x) \ge k.$$

Since $f_s(v) > 0$, this inequality chain shows that $v \in N[u]$. Thus v is adjacent to u. Since v and u are arbitrary vertices of G, it follows that G is isomorphic to the complete graph K_n. □

Theorem 2.3. *If $k \geq j \geq 1$ are two integers, and G is a graph of order n, then*

$$\gamma_{(j,k)}(G) + d_{(j,k)}(G) \leq nj + 1.$$

Proof. If $j(\delta(G)+1) < k$ then $\gamma_{(j,k)}(G)+d_{(j,k)}(G) \leq nj$ by (2), (3) and the fact $\gamma(G) \leq n$. Let $j(\delta(G)+1) \geq k$. Applying Theorem 2.1, we obtain

$$\gamma_{(j,k)}(G) + d_{(j,k)}(G) \leq \frac{jn}{d_{(j,k)}(G)} + d_{(j,k)}(G).$$

Inequality (1) yields to $d_{(j,k)}(G) \geq 1$, and Theorem 2.2 implies that $d_{(j,k)}(G) \leq \frac{jn}{k}$. Using these inequalities, and the fact that the function $g(x) = x + (jn)/x$ is decreasing for $1 \leq x \leq \sqrt{jn}$ and increasing for $\sqrt{jn} \leq x$, we obtain

$$\gamma_{(j,k)}(G) + d_{(j,k)}(G) \leq \max\left\{jn+1, \frac{jn}{k} + k\right\} = nj + 1,$$

and this is the desired bound. □

Corollary 2.4. [2] *If G is a graph of order $n \geq 1$, then $\gamma(G) + d(G) \leq n + 1$.*

Theorem 2.5. *If $k \geq j \geq 1$ are two integers such that $k \geq 2$, and G is a graph of order n and $d_{(j,k)}(G) \geq 2$, then*

$$\gamma_{(j,k)}(G) + d_{(j,k)}(G) \leq \frac{jn}{2} + 2.$$

Proof. Theorem 2.1 implies that

$$\gamma_{(j,k)}(G) + d_{(j,k)}(G) \leq \gamma_{(j,k)}(G) + \frac{jn}{\gamma_{(j,k)}(G)}.$$

The hypothesis $d_{(j,k)}(G) \geq 2$ and Theorem 2.2 lead to $2k \leq jn$. In addition, it follows from Theorem 2.1 that $k \leq \gamma_{(j,k)}(G) \leq jn/2$. Using these inequalities, the condition $k \geq 2$, and the fact that the function $g(x) = x + (jn)/x$ is decreasing for $-\sqrt{jn} \leq x \leq \sqrt{jn}$ and increasing for $\sqrt{jn} \leq x \leq jn/2$, we obtain

$$\begin{aligned}
\gamma_{(j,k)}(G) + d_{(j,k)}(G) &\leq \max\left\{k + \frac{jn}{k}, \frac{jn}{2} + \frac{2jn}{jn}\right\} \\
&= \max\left\{k + \frac{jn}{k}, \frac{jn}{2} + 2\right\} \\
&= \frac{jn}{2} + 2,
\end{aligned}$$

and this is the desired bound. □

Corollary 2.6. [4] *If G is a graph with $\delta(G) \geq k-1 \geq 1$ and $d_{\times k} \geq 2$, then $\gamma_{\times k}(G) + d_{\times k}(G) \leq \lfloor \frac{n}{2} \rfloor + 2$.*

Theorem 2.7. *If $k \geq j \geq 1$ are two integers, and G is a graph of order n, then*

$$d_{(j,k)}(G) \leq \frac{j(\delta(G)+1)}{k}.$$

Moreover, if $d_{(j,k)}(G) = \frac{j(\delta(G)+1)}{k}$, then for each function of any $(j,k)D$ family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u \in N[v]} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = j$ for every $u \in N[v]$.

Proof. If $j(\delta(G)+1) < k$ then by (3), $d_{(j,k)}(G) = 0 < \frac{j(\delta(G)+1)}{k}$. Let $j(\delta(G)+1) \geq k$ and let $\{f_1, f_2, \ldots, f_d\}$ be a (j,k)D family on G such that $d = d_{(j,k)}(G)$, and let v be a vertex of minimum degree $\delta(G)$. Since $\sum_{u \in N[v]} f_i(u) \geq k$ for all $i \in \{1, 2, \ldots, d\}$, we obtain

$$\begin{aligned} kd &\leq \sum_{i=1}^{d} \sum_{u \in N[v]} f_i(u) \\ &= \sum_{u \in N[v]} \sum_{i=1}^{d} f_i(u) \\ &\leq \sum_{u \in N[v]} j = j(\delta(G)+1), \end{aligned}$$

and this leads to the desired bound.

If $d_{(j,k)}(G) = \frac{j(\delta(G)+1)}{k}$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. □

Corollary 2.8. [4] *Let k be a positive integer, and G have $\delta(G) \geq k-1$. Then $d_{\times k}(G) \leq \lfloor \frac{\delta(G)+1}{k} \rfloor$.*

Corollary 2.9. [2] *For any graph G, $d(G) \leq \delta(G) + 1$.*

As a further application of Theorem 2.7, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.10. *For every graph G of order n with $j(\delta(G)+1) \geq k$ and $j(\delta(\overline{G})+1) \geq k$,*

$$d_{(j,k)}(G) + d_{(j,k)}(\overline{G}) \leq \frac{j(n - \Delta(G) + \delta(G) + 1)}{k} \leq \frac{j(n+1)}{k}.$$

If $d_{(j,k)}(G) + d_{(j,k)}(\overline{G}) = j(n+1)/k$, then k is a devisor of $j(\delta(G)+1)$ and of $j(\delta(\overline{G})+1)$, and G is a regular graph.

Proof. It follows from Theorem 2.7 that

$$\begin{aligned} d_{(j,k)}(G) + d_{(j,k)}(\overline{G}) &\leq \frac{j(\delta(G)+1)}{k} + \frac{j(\delta(\overline{G})+1)}{k} \\ &= \frac{j(n-\Delta(G)+\delta(G)+1)}{k} \\ &\leq \frac{j(n+1)}{k}. \end{aligned}$$

If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and we obtain as above the better bound

$$d_{(j,k)}(G) + d_{(j,k)}(\overline{G}) \leq \frac{jn}{k}.$$

If k is not a devisor of $j(\delta(G)+1)$ or of $j(\delta(\overline{G})+1)$, then

$$d_{(j,k)}(G) + d_{(j,k)}(\overline{G}) < \frac{j(\delta(G)+1)}{k} + \frac{j(\delta(\overline{G})+1)}{k} \leq \frac{j(n+1)}{k},$$

and the proof is complete. □

If G is isomorphic to the complete graph of order n, then $d_{\{k\}}(G) = n$ and $d_{\{k\}}(\overline{G}) = 1$. Thus $d_{\{k\}}(K_n) + d_{\{k\}}(\overline{K_n}) = n + 1$. This example shows that Theorems 2.10 is sharp at least for $j = k$.

Again some results concerning the k-tuple domatic number and the domatic number follow as corollaries.

Corollary 2.11. [4] *For a graph G with $\delta(G), \delta(\overline{G}) \geq k-1$,*

$$d_{\times k}(G) + d_{\times k}(\overline{G}) \leq \frac{n-\Delta(G)+\delta(G)+1}{k}.$$

Corollary 2.12. [2] *If G is a graph of order $n \geq 1$, then $d(G) + d(\overline{G}) \leq n + 1$.*

References

[1] G. S. Domke, S. T. Hedetniemi, R. C. Laskar, and G. H. Fricke, *Relationships between integer and fractional parameters of graphs*, In Y. Alavi, G. Chartrand, O.R. Oellermann and J. Schwenk, (Eds.), Graph Theory, Combinatorics, and Applications, Proceedings of the Sixth Quadrennial International Conference on the Theory and Ap- plications of Graphs, Vol. 1, pages 371-387, (Kalamazoo, MI 1988), Wiley Publications, 1991.

[2] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks*, **7**(1977), 247-261.

[3] F. Harary and T. W. Haynes, Nordhaus-Gaddum inequalities for domination in graphs, *Discrete Math.*, **155**(1996), 99-105.

[4] F. Harary and T. W. Haynes, The k-tuple domatic number of a graph, *Math. Slovaca*, **48**(1998), 161-166.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in graphs*, Marcel Dekker, Inc., New York, 1998.

[6] R. R. Rubalcaba and P. J. Slater, Efficient (j, k)-domination, *Discuss. Math. Graph Theory*, **27**(2007), 409-423.

[7] R. R. Rubalcaba and P. J. Slater, A note on obtaining k dominating sets from a k-dominating function on a tree, *Bull. Inst. Combin. Appl.*, **51**(2007), 47-54.

[8] D. Meierling, S. M. Sheikholeslami and L. Volkmann, The $\{k\}$-domatic number of a graph, *Aequationes Math.*, **82**(2011), 25-34.

[9] X. Hou and Y. Lu, On the $\{k\}$-domination number of Cartesian products of graphs, *Discrete Math.*, **309**(2009), 3413-3419.

[10] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.