ON THE ROMAN \( k \)-BONDAGE NUMBER OF A GRAPH

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Abstract

A Roman dominating function on a graph \( G = (V, E) \) is a function \( f : V \to \{0, 1, 2\} \) such that every vertex \( v \in V \) with \( f(v) = 0 \) has at least one neighbor \( u \in V \) with \( f(u) = 2 \). The weight of a Roman dominating function is the value \( f(V(G)) = \sum_{u \in V(G)} f(u) \). The minimum weight of a Roman dominating function on a graph \( G \) is called the Roman domination number, denoted by \( \gamma_R(G) \). The Roman bondage number \( b_R(G) \) of a graph \( G \) with maximum degree at least two is the minimum cardinality of all sets \( E' \subseteq E(G) \) for which \( \gamma_R(G - E') > \gamma_R(G) \).

In this note we first present sharp bounds for \( b_R(G) \) and then we initiate the study of the Roman \( k \)-bondage number in graphs. Some of our results extend those given by Jafari Rad and Volkmann in 2011 for the Roman bondage number.

Keywords: Roman domination number, Roman bondage number, Roman \( k \)-domination number, Roman \( k \)-bondage number.

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1. Introduction

In this paper, \( G \) is a simple graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The order \( |V| \) of \( G \) is denoted by \( n = n(G) \). For every vertex \( v \in V \), the open neighborhood \( N(v) \) is the set \( \{u \in V \mid uv \in E\} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \in V \) is \( \text{deg}_G(v) = \text{deg}(v) = |N(v)| \). The minimum and maximum degree of a graph \( G \) are denoted by \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \), respectively. The open neighborhood of a set \( S \subseteq V \) is the set \( N(S) = \bigcup_{v \in S} N(v) \), and the closed neighborhood of \( S \) is the set \( N[S] = N(S) \cup S \). The complement \( \overline{G} \) of \( G \) is the simple graph whose vertex set is \( V \) and whose edges are the pairs of nonadjacent vertices of \( G \). We write \( K_n \) for the
complete graph of order \( n \) and \( C_n \) for a cycle of length \( n \). The reader is referred to [6, 17] for any terminology and notation not defined here.

Let \( k \) be a positive integer. A subset \( S \) of vertices of \( G \) is a \( k \)-dominating set if \( |N(v) \cap S| \geq k \) for every \( v \in V - S \). The \( k \)-domination number \( \gamma_k(G) \) is the minimum cardinality of a \( k \)-dominating set of \( G \). The \( k \)-bondage number of a graph \( G \) with \( \Delta \geq k \) is the minimum cardinality among all sets of edges \( B \subseteq E \) for which \( \gamma_k(G - B) > \gamma_k(G) \). The \( k \)-bondage number was introduced by Lu and Xu [12].

Let \( k \geq 1 \) be an integer. Following Kämmerling and Volkmann [10], a Roman \( k \)-dominating function on a graph \( G \) is a labeling \( f : V \to \{0, 1, 2\} \) such that every vertex with label 0 has at least \( k \) neighbors with label 2. The weight of a Roman \( k \)-dominating function is the value \( f(V) = \sum_{u \in V(G)} f(u) \). The minimum weight of a Roman \( k \)-dominating function on a graph \( G \) is called the Roman \( k \)-domination number, denoted by \( \gamma_{kR}(G) \). Note that the Roman 1-domination number \( \gamma_{1R}(G) \) is the usual Roman domination number \( \gamma_R(G) \). A \( \gamma_{kR}(G) \)-function is a Roman \( k \)-dominating function on \( G \) with weight \( \gamma_{kR}(G) \).

A Roman \( k \)-dominating function \( f : V \to \{0, 1, 2\} \) can be represented by the ordered partition \((V_0, V_1, V_2)\) (or \((V_0^f, V_1^f, V_2^f)\) to refer to \( f \)) of \( V \), where \( V_i = \{v \in V \mid f(v) = i\} \).

In this representation, its weight is \( \omega(f) = |V_1| + 2|V_2| \). Since \( V_1^f \cup V_2^f \) is a \( k \)-dominating set when \( f \) is an RkDF, and since placing weight 2 at the vertices of a \( k \)-dominating set yields an RkDF, in [10], it was observed that

\[
\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G) \tag{1}
\]

The definition of the Roman dominating function was given implicitly by Stewart [16] and ReVelle and Rosing [15]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [2] as well as Chambers, Kinnersley, Prince and West [1] have given a lot of results on Roman domination. For more information on Roman domination we refer the reader to [3, 4, 5, 7, 8, 9, 11].

Let \( k \) be a positive integer, and let \( G \) be a graph with maximum degree at least two. The Roman \( k \)-bondage number \( b_{kR}(G) \) of \( G \) is the minimum cardinality of all sets \( E' \subseteq E \) for which \( \gamma_{kR}(G - E') > \gamma_{kR}(G) \). When \( k = 1 \), the Roman \( k \)-bondage number \( b_{kR}(G) \) is the usual Roman bondage number \( b_R(G) \) which was introduced by Jafari Rad and Volkmann in [13], and has been further studied for example in [14].

Our purpose in this paper is to initiate the study of the Roman \( k \)-bondage number in graphs. We first present some general upper bounds for Roman bondage number and then we study basic properties and bounds for the Roman \( k \)-bondage number of a graph. In addition, we determine the Roman \( k \)-bondage number of some classes of graphs.

We make use of the following observations and results for our investigations.

**Observation 1.1.** Let \( k \) be a positive integer, and let \( G \) be a graph of order \( n \) with \( \gamma_{kR}(G) = n \). Then for any \( E' \subseteq E \) we have \( \gamma_{kR}(G) = n = \gamma_{kR}(G - E') \).

**Observation 1.2.** Let \( k \) be a positive integer, and let \( G \) be a graph of order \( n \). If \( n \leq 2k \) or \( \Delta < k \), then \( \gamma_{kR}(G) = n \).
Proposition A. [10] If $G$ is a graph of order $n$ and maximum degree $\Delta = k$, then $\gamma_{kR}(G) = n$.

Proposition B. [10] If $G$ is a graph of order $n$ with at most one cycle, then $\gamma_{kR}(G) = n$ when $k \geq 2$.

Proposition C. [10] Let $G$ be a graph of order $n$. Then $\gamma_{kR}(G) < n$ if and only if $G$ contains a bipartite subgraph $H$ with bipartition $X, Y$ such that $|X| > |Y| \geq k$ and $d_H(v) \geq k$ for each $v \in X$.

Proposition D. [13] For $n \geq 3$, $b_{R}(K_n) = \lceil n/2 \rceil$.

Proposition E. [13] If $G$ is a graph, and $uvw$ a path of length 2 in $G$, then

$$b_{R}(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 3.$$ 

If $u$ and $w$ are adjacent, then

$$b_{R}(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 4.$$ 

Regarding Observations 1.1, 1.2 and Propositions A and B, in the study of the Roman $k$-bondage $b_{kR}(G)$ we will assume that $\gamma_{kR}(G) < n$, $n > 2k$, $\Delta > k$ and that $G$ is not a tree or unicyclic graph.

We start with the following observations and properties.

Observation 1.3. Let $G$ be a graph. Suppose $q$ edges can be removed from $G$ to give a graph $H$ with $b_{kR}(H) = 1$. Then $b_{kR}(G) \leq q + 1$.

Observation 1.4. Let $G$ be a graph of order $n$ with $\gamma_{kR}(G) < n$. Assume that $H$ is a spanning subgraph of $G$ with $\gamma_{kR}(H) = \gamma_{kR}(G)$. If $K = E(G) - E(H)$, then $b_{kR}(H) \leq b_{kR}(G) \leq b_{kR}(H) + |K|$.

If $P_n$ and $C_n$ are the path and cycle of order $n$, then it was shown in [2] that $\gamma_{R}(P_n) = \gamma_{R}(C_n) = \lceil 2n/3 \rceil$. In addition, we find in [13] for $n \geq 3$:

1. $b_{R}(P_n) = 2$ if $n \equiv 2 \pmod{3}$ and $b_{R}(P_n) = 1$ otherwise,

2. $b_{R}(C_n) = 3$ if $n \equiv 2 \pmod{3}$ and $b_{R}(C_n) = 2$ otherwise.

Using these results and Observation 1.4, we obtain:

Corollary 1.5. Let $G$ be a graph of order $n \geq 3$.

1. If $G$ has a Hamiltonian path and $\gamma_{R}(G) = \lceil 2n/3 \rceil$, then $b_{R}(G) \geq 2$ if $n \equiv 2 \pmod{3}$,

2. If $G$ is Hamiltonian with $\gamma_{R}(G) = \lceil 2n/3 \rceil$, then $b_{R}(G) \geq 2$ and $b_{R}(G) \geq 3$ if $n \equiv 2 \pmod{3}$. 

Observation 1.6. If a graph $G$ has a vertex $v$ such that $\gamma_{kR}(G - v) \geq \gamma_{kR}(G)$, then $b_{kR}(G) \leq \deg(v) \leq \Delta$.

Observation 1.7. If a graph $G$ has a vertex $v$ such that every $\gamma_{kR}(G)$-function assigns 2 to $v$, then $b_{kR}(G) \leq \deg(v) \leq \Delta$.

Proof. If $E_v$ is the set of edges incident with $v$, then we will show that $\gamma_{kR}(G - E_v) > \gamma_{kR}(G)$. Suppose to the contrary that $\gamma_{kR}(G - E_v) \leq \gamma_{kR}(G)$. Then $\gamma_{kR}(G - E_v) = \gamma_{kR}(G - v) + 1$ and hence $\gamma_{kR}(G) \geq \gamma_{kR}(G - v) + 1$. Since $\gamma_{kR}(G) \leq \gamma_{kR}(G - v) + 1$, we obtain $\gamma_{kR}(G) = \gamma_{kR}(G - v) + 1$, a contradiction to the hypothesis that every $\gamma_{kR}(G)$-function assigns 2 to $v$.

2. Bounds on the Roman bondage number

In this section we establish bounds on the Roman bondage number of a graph that are independent of the graph structure.

Theorem 2.1. If $G$ is a connected graph of order $n \geq 3$, then

$$b_R(G) \leq n - 1.$$  

Furthermore, this bound is sharp for $K_3$.

Proof. Let $u$ and $v$ be two adjacent vertices with $\deg(u) \leq \deg(v)$. If $b_R(G) \leq \deg(u) \leq n - 1$, then we are done. Suppose to the contrary that $b_R(G) > \deg(u)$.

Let $E_u$ denote the set of edges incident with $u$. Then $\gamma_R(G - E_u) = \gamma_R(G)$ and $\gamma_R(G - u) = \gamma_R(G) - 1$. Assume that $\mathcal{F} = \{f = (V_0^f, V_1^f, V_2^f) \mid f$ is a $\gamma_R(G - u)$-function $\}$ and $V_2 = \cup_{f \in \mathcal{F}} V_2^f$. Then $u$ is adjacent to no vertex of $V_2$ in $G$. Hence $|E_u| \leq n - 1 - |V_2|$ and $v \notin V_2$. Let $F_v$ denote the set of edges from $v$ to a vertex in $V_2$. If $\gamma_R(G - u - F_v) > \gamma_R(G - u)$ or equivalently $\gamma_R(G - u - F_v) > \gamma_R(G) - 1$, then $\gamma_R(G - (E_u \cup F_v)) > \gamma_R(G)$, and we see that

$$b_R(G) \leq |E_u \cup F_v| \leq (n - 1 - |V_2|) + |V_2| = n - 1.$$

So we assume that $\gamma_R(G - u - F_v) = \gamma_R(G - u)$. Then $\gamma_R(G - u - v) = \gamma_R(G - u) - 1$. Since $G$ is connected of order $n \geq 3$ and since $\deg(u) \leq \deg(v)$, we may assume that $w \in N(v) - \{u\}$. Let $\mathcal{F}' = \{f = (V_0^f, V_1^f, V_2^f) \mid f$ is a $\gamma_R(G - u - v)$-function $\}$, and let $V_2' = \cup_{f \in \mathcal{F}'} V_2^f$. Then $u$ and $v$ are not adjacent to any vertex of $V_2'$ in $G$. Let $F_w$ denote the set of edges from $w$ to a vertex in $V_2'$ and so $w \notin V_2'$.

If $w \in V_1^h$ for some $h \in \mathcal{F}'$, then $\gamma_R(G - \{u, v, w\}) = \gamma_R(G - \{u, v\}) - 1 = \gamma_R(G) - 3$ and obviously the function $g : V(G) \to \{0, 1, 2\}$ defined by

$$g(u) = g(v) = 0, g(v) = 2 \text{ and } g(x) = h(x) \text{ for } x \in V(G) - \{u, v, w\},$$
is a Roman dominating function on $G$ of weight less than $\gamma_R(G)$ which is a contradiction. Therefore $\gamma_R(G-u-v-F_u) > \gamma_R(G-u-v)$ or equivalently $\gamma_R(G-u-v-F_u) > \gamma_R(G)-2$. Thus $\gamma_R(G-(E_u \cup F_v \cup F_w)) > \gamma_R(G)$ and we see that

$$b_R(G) \leq |E_u \cup F_v \cup F_w| \leq (n - 1 - |V'_2|) + |V'_2| = n - 1,$$

and the proof is complete.

By a closer look at the proof of Theorem 2.1 we have the following result improving Proposition E.

**Corollary 2.2.** If $G$ is a connected graph of order $n \geq 3$ and $uvw$ a path of length 2 in $G$, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - |N(w) \cap (N(u) \cup N(v)) - \{u, v\}| - 3.$$

If $u$ and $w$ are adjacent, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - |N(w) \cap (N(u) \cup N(v)) - \{u, v\}| - 4.$$

The next result presents an upper bound on the Roman bondage number that involves the maximum degree. This bound also indicates a relationship between the Roman bondage number and the Roman domination number. If $\gamma_R(G) = 2$, then obviously $b_R(G) \leq \delta(G)$. So we assume that $\gamma_R(G) \geq 3$.

**Theorem 2.3.** If $G$ is a connected graph of order $n \geq 4$ with Roman domination number $\gamma_R(G) \geq 3$, then

$$b_R(G) \leq (\gamma_R(G)-2)\Delta(G) + 1.$$

**Proof.** The proof is by induction on $\gamma_R(G)$. First suppose $\gamma_R(G) = 3$. Let $u$ be a vertex of maximum degree in $G$ and let $E_u$ denote the set of edges incident with $u$. If $\gamma_R(G-E_u) > \gamma_R(G)$, then $b_R(G) \leq |E_u| = \deg(u)$ and hence $b_R(G) \leq \Delta(G)$. Assume that $\gamma_R(G-E_u) = \gamma_R(G)$ or equivalently $\gamma_R(G-u) = \gamma_R(G) - 1 = 2$. Since $n \geq 3$, there is a vertex $v$ that is adjacent with every vertex of $G$ but $u$. Thus $\deg_{G}(v) = \Delta$ also, and $u$ is adjacent with every vertex of $G$ except $v$. If there is an edge $e$ incident with $v$ such that $\gamma_R(G-u-e) > \gamma_R(G-u)$, then obviously $b_R(G) \leq \deg(u) + 1 \leq \Delta + 1$ and we are done. Assume that the removal from $G-u$ of any one edge incident with $v$ again leaves a graph with Roman domination number 2. It follows that there is a vertex $w \neq v$ that is adjacent to every vertex of $G-u$. Since the vertex $v$ is the only vertex of $G$ that is not adjacent with $u$, we deduce that $w$ must be adjacent in $G$ with $u$. This however implies that $\gamma_R(G) = 2$ which is a contradiction.

Now assume that the statement is true for any graph of order $n \geq 4$ with Roman domination number $3 \leq \gamma_R(G) \leq k$. Let $G$ be a graph of order $n \geq 4$ with $\gamma_R(G) = k + 1$. Suppose to the contrary that $b_R(G) > (\gamma_R(G)-2)\Delta(G) + 1$. Then for any vertex $u$ of
Then obviously $f \in \mathbb{R}$. Thus we determine the Roman $k$-bondage number.

**Theorem 3.1.** Let $k$ be a positive integer and let $G$ be a graph with $\Delta(G) \geq k + 1$. Then $b_{kR}(G) = 1$ if and only if $G$ has an edge $xy$ satisfying either $x \in V_2$ and $y \in V_0 \cap PN_k(x, V_2)$ or $y \in V_2$ and $x \in V_0 \cap PN_k(y, V_2, G)$ for any $\gamma_{kR}(G)$-function $f = (V_0, V_1, V_2)$.

**Proof.** Assume that $b_{kR}(G) = 1$. Then there is an edge, denoted by $xy$, in $G$ such that $\gamma_{kR}(G - xy) > \gamma_{kR}(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G)$-function. If $x \in V_1$ or $y \in V_1$, then obviously $f$ is a $\gamma_{kR}(G - xy)$-function which is a contradiction. If $x, y \in V_0$ or $x, y \in V_2$, then clearly $f$ is a $\gamma_{kR}(G - xy)$-function which is a contradiction again. Thus we may assume, without loss of generality, that $x \in V_2$ and $y \in V_0$. Then $y$ has at least $k$ neighbors in $V_2$. If $|V_2 \cap N_G(y)| > k$ then $f$ is a $\gamma_{kR}(G - xy)$-function, a contradiction. Thus $|V_2 \cap N_G(y)| = k$ and hence $y \in V_0 \cap PN_k(x, V_2)$.

Conversely, we assume that the edge $xy$ of $G$ satisfies either $x \in V_2$ and $y \in V_0 \cap PN_k(x, V_2)$ or $y \in V_2$ and $x \in V_0 \cap PN_k(y, V_2, G)$ for any $\gamma_{kR}(G)$-function $f = (V_0, V_1, V_2)$. Then we only need to prove $\gamma_{kR}(G - xy) > \gamma_{kR}(G)$. Suppose to the contrary that $\gamma_{kR}(G - xy) = \gamma_{kR}(G)$.

Let $g = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G - xy)$-set. Since $\gamma_{kR}(G - xy) = \gamma_{kR}(G)$, $g$ is a $\gamma_{kR}(G)$-set. Then we may assume, without loss of generality, that $x \in V_2^g$ and $y \in V_0^g \cap PN_k(x, V_2^g, G)$. This implies that $|V_2^g \cap N_{G - xy}(y)| = k - 1$ which is a contradiction. This completes the proof.

**Corollary 3.2.** Let $k$ be a positive integer, and let $G$ be a graph of order $n$ with $\gamma_{kR}(G) < n$. If $G$ has a unique $\gamma_{kR}(G)$-function, then $b_{kR}(G) = 1$.

**Proof.** Let $f = (V_0^f, V_1^f, V_2^f)$ be the $\gamma_{kR}(G)$-function. If there exists a vertex $u \in V_0^f$ with $v \in V_2^f \cap N(u)$ such that $u \in PN_k(v, V_2, G)$, then the result follows from Theorem 3.1.
Then $b$ is a RkDF of the condition 1 is fulfilled. For any $\gamma$-function $f = (V_0, V_1, V_2)$ and there exists $\gamma_k(G)$-functions $f_i (i = 1, 2)$ such that $v_i \in V_0 \cap PN_k(u_i, V_2^f, G)$ for each $i$.

Proof. Assume that $b_k(G) = 2$. Then there are two edges, denoted by $u_1v_1, u_2v_2$, in $G$ such that $\gamma_k(G - \{u_1v_1, u_2v_2\}) > \gamma_k(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_k(G)$-function. We consider two cases.

**Case 1.** $\{u_1v_1\} \cap \{u_2v_2\} \neq \emptyset$.

Then we may assume, without loss of generality, that $u_1 = u_2$. If $u_1 \in V_1$, then obviously $f$ is a $\gamma_k(G - \{u_1v_1, u_2v_2\})$-dominating function which is a contradiction. Let $u \in V_2$. If $v_1 \not\in V_0 \cap PN_{k+1}(u_1, V_2, G)$ and $v_2 \not\in V_0 \cap PN_{k+1}(u_1, V_2, G)$, then $f$ is a $\gamma_k(G - \{u_1v_1, u_2v_2\})$-dominating function, a contradiction. Thus $v_1 \in V_0 \cap PN_{k+1}(u_1, V_2, G)$ or $v_2 \in V_0 \cap PN_{k+1}(u_1, V_2, G)$. Since $b_k(G) = 2$, it follows from Theorem 3.1 that there exist $\gamma_k(G)$-functions $f_i (i = 1, 2)$ such that $v_i \in V_0 \cap PN_{k+1}(u_1, V_2^f, G)$ for each $i$. Therefore the condition 1 is fulfilled.

Finally, let $u_1 \in V_0$. If $v_1, v_2 \in V_2$ and $u_1 \in PN_{k+1}(v_1, V_2, G)$ and $u_1 \in PN_{k+1}(v_2, V_2, G)$, then the condition 2 is fulfilled. Let $\{v_1, v_2\} \not\subseteq V_2$, $u_1 \not\in PN_{k+1}(v_1, V_2, G)$ or $u_1 \not\in PN_{k+1}(v_2, V_2, G)$. If $v_1, v_2 \not\in V_2$, $v_1 \in V_2$ and $u_1 \not\in PN_k(v_1, V_2, G)$ or $v_2 \in V_2$ and $u_1 \not\in PN_k(v_2, V_2, G)$, then obviously $f$ is a $\gamma_k(G - \{u_1v_1, u_2v_2\})$-dominating function, a contradiction. Thus we may assume $v_1 \in V_2$ and $u_1 \in PN_k(v_1, V_2, G)$ or $v_2 \in V_2$ and $u_1 \in PN_k(v_1, V_2, G)$. Since $b_k(G) = 2$, it follows from Theorem 3.1 that there exist
\(\gamma_{kR}(G)\)-functions \(f_i\) \((i = 1, 2)\) such that \(v_i \in V_2^f\) and \(u_1 \in PN_k(v_i, V_2^f, G)\). Hence the condition 3 holds.

**Case 2.** \(\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset\).

Then the edges \(u_1v_1\) and \(u_2v_2\) are independent. If \(\{u_i, v_i\} \cap V_2 = \emptyset\) for \(i = 1, 2\), then obviously \(f\) is a \(\gamma_{kR}(G - \{u_1v_1, u_2v_2\})\)-function which is a contradiction. Thus \(\{u_i, v_i\} \cap V_2 \neq \emptyset\) for some \(i\). Let \(\{u_1, v_1\} \cap V_2 \neq \emptyset\) (the proof of the case \(\{u_2, v_2\} \cap V_2 \neq \emptyset\) is similar). If \(\{u_1, v_1\} \subseteq V_2\), then \(f\) is a \(\gamma_{kR}(G - \{u_1v_1\})\)-function. If \(\{u_2, v_2\} \cap V_1 \neq \emptyset\), \(\{u_2, v_2\} \subseteq V_0\) or \(\{u_2, v_2\} \subseteq V_2\), then \(f\) is a \(\gamma_{kR}(G - \{u_1v_1, u_2v_2\})\)-function which is a contradiction. Therefore we may assume, without loss of generality, that \(v_2 \in V_2\) and \(u_2 \in V_0\). Since \(\gamma_{kR}(G - \{u_1v_1, u_2v_2\}) > \gamma_{kR}(G - \{u_1v_1\}) = \gamma_{kR}(G)\), we must have \(u_2 \in PN_k(v_2, V_2, G)\).

If \(v_2 \in V_2\) and \(u_2 \in PN_k(v_2, V_2, G)\) or \(u_2 \in V_0\) and \(u_2 \in PN_k(v_2, V_2, G)\) for every \(\gamma_{kR}(G)\)-function, then \(b_{kR}(G) = 1\) by Theorem 3.1, a contradiction. Thus there is a \(\gamma_{kR}(G)\)-function, say \(g\), such that \(v_2 \not\in V_2\) or \(u_2 \not\in PN_k(v_2, V_2, G)\) and \(u_2 \not\in V_0\) or \(v_2 \not\in PN_k(v_2, V_2, G)\). Then \(g\) is a \(\gamma_{kR}(G - u_2v_2)\)-function and hence \(\gamma_{kR}(G) = \gamma_{kR}(G - u_2v_2)\). An argument similar to that described above \(v_1 \in V_2\) and \(u_1 \in PN_k(v_1, V_2, G)\) or \(u_1 \in V_2\) and \(v_1 \in PN_k(u_1, V_2, G)\). Thus the condition 3 holds.

Conversely, we assume that \(G\) has two edges \(u_1v_1, u_2v_2\) satisfying one of the above conditions. It is easy to see that \(\gamma_{kR}(G - \{u_1v_1, u_2v_2\}) > \gamma_{kR}(G)\). This completes the proof.

**Theorem 3.4.** Let \(k\) be a positive integer, and let \(G\) be a graph of order \(n > 2k\). If \(G\) has exactly \(m \geq k\) vertices of degree \(n - 1\), then

\[
b_{kR}(G) = \left\lfloor \frac{m - k + 1}{2} \right\rfloor.
\]

**Proof.** Since \(n > 2k\) and \(m \geq k\), \(\gamma_{kR}(G) = 2k\). Suppose that \(E' \subseteq E(G)\) is an arbitrary subset of edges such that \(|E'| < \left\lfloor \frac{m - k + 1}{2} \right\rfloor\). It is clear that \(G - E'\) has at least \(k\) vertices of degree \(n - 1\) and hence \(\gamma_{kR}(G - E') = 2k\). It follows that \(b_{kR}(G) \geq \left\lfloor \frac{m - k + 1}{2} \right\rfloor\).

Let \(S = \{v_1, v_2, \ldots, v_m\}\) be the set of vertices of degree \(n - 1\) in \(G\). Then the subgraph \(G[S]\) is the complete graph \(K_m\). If \(m - k + 1\) is even, then let \(H_1\) be the graph obtained from \(G\) by removing \(\frac{m - k + 1}{2}\) independent edges from \(G[S]\). Then \(H_1\) has exactly \(k - 1\) vertices of degree \(n - 1\). This implies that \(\gamma_{kR}(H_1) \geq 2k + 1\). Hence \(b_{kR}(G) \leq \left\lfloor \frac{m - k + 1}{2} \right\rfloor\).

If \(m - k + 1\) is odd, then let \(H_2\) be the graph obtained from \(G\) by removing \(\frac{m - k + 2}{2}\) independent edges from \(G[S]\). Then \(H_2\) has exactly \(k - 2\) vertices of degree \(n - 1\). Thus \(\gamma_{kR}(H_2) \geq 2k + 1\). This implies that \(b_{kR}(G) \leq \left\lfloor \frac{m - k + 1}{2} \right\rfloor\). Combining the obtained inequalities, we have \(b_{kR}(G) = \left\lfloor \frac{m - k + 1}{2} \right\rfloor\), and the proof is complete.

The special case \(k = 1\) of the next result can be found in [13].
Corollary 3.5. If $n > 2k$, then $b_{kR}(K_n) = \lceil \frac{n-k+1}{2} \rceil$.

If $G$ is isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p$, then by Proposition C, $\gamma_{kR}(G) = p + q = n$ when $p < k$ or $q = p = k$. Now we determine the Roman $k$-domination number for complete bipartite graphs $K_{p,q}$ with $q \geq p \geq k$ and $p + q \geq 2k + 1$.

Theorem 3.6. Let $k$ be a positive integer, and let $G$ be isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p \geq k$ such that $n = p + q \geq 2k + 1$. Then

$$\gamma_{kR}(G) = \min\{k + p, 4k\}.$$

Proof. Let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$ be the partite sets of $G$.

If $p \leq 3k$, then define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(u_1) = f(u_2) = \ldots = f(u_k) = 2$, $f(u_{k+1}) = f(u_{k+2}) = \ldots = f(u_p) = 1$ and $f(v_i) = 0$ for $1 \leq i \leq q$. Then $f$ is a Roman $k$-dominating function of weight $2k + p - k = k + p$ and thus $\gamma_{kR}(G) \leq k + p < p + q = n$.

If $p \geq 3k + 1$, then define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(u_1) = g(u_2) = \ldots = g(u_k) = 2$, $g(v_1) = g(v_2) = \ldots = g(v_k) = 2$, and $g(u_i) = g(v_j) = 0$ for $k + 1 \leq i \leq p$ and $k \leq j \leq q$. Then $g$ is a Roman $k$-dominating function of weight $4k$ and thus $\gamma_{kR}(G) \leq 4k < p + q = n$.

Now let $f = (V_0, V_1, V_2)$ be any $\gamma_{kR}(G)$-function. If $V_0 = \emptyset$, then $\gamma_{kR}(G) = n$, a contradiction. Hence we can assume that $|V_0| \geq 1$. If $V_0 \cap X \neq \emptyset$ and $V_0 \cap Y \neq \emptyset$, then $|V_2 \cap Y| \geq k$ and $|V_2 \cap X| \geq k$ and thus $\gamma_{kR}(G) \geq 4k$. If $V_0 \cap X \neq \emptyset$ and $V_0 \cap Y = \emptyset$, then $|V_2 \cap Y| = k$ and $|V_1 \cap Y| = q - k$ and thus $\gamma_{kR}(G) \geq 2k + q - k = k + q$. If $V_0 \cap X = \emptyset$ and $V_0 \cap Y \neq \emptyset$, then $|V_2 \cap X| = k$ and $|V_1 \cap X| = p - k$ and thus $\gamma_{kR}(G) \geq 2k + p - k = k + p$.

Combining the last three inequalities, we see that

$$\gamma_{kR}(G) \geq \min\{k + p, k + q, 4k\} = \min\{k + p, 4k\},$$

and altogether, we obtain the desired result.

Theorem 3.7. Let $k$ be a positive integer, and let $G$ be isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p \geq k$ such that $n = p + q \geq 2k + 1$. Then

$$b_{kR}(G) = p - k + 1.$$

Proof. Let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$ be the partite sets of $G$. Suppose that $E' \subseteq E(G)$ is an arbitrary edge set with $|E'| \leq p - k$, and let $G' = G - E'$. Then $X \cap V(G')$ contains at least $k$ vertices, say $u_1, u_2, \ldots, u_k$ of degree $q$ and $Y \cap V(G')$ contains at least $k$ vertices, say $v_1, v_2, \ldots, v_k$ of degree $p$.

If $p \leq 3k$, then define $f : V(G') \rightarrow \{0, 1, 2\}$ by $f(u_1) = f(u_2) = \ldots = f(u_k) = 2$, $f(u_{k+1}) = f(u_{k+2}) = \ldots = f(u_p) = 1$ and $f(v_i) = 0$ for $1 \leq i \leq q$. Then $f$ is a Roman $k$-dominating function on $G'$ of weight $2k + p - k = k + p$ and thus $\gamma_{kR}(G') \leq k + p$. It follows from Theorem 3.6 that $\gamma_{kR}(G') = \gamma_{kR}(G)$ and therefore $b_{kR}(G) \geq p - k + 1$.

If $p \geq 3k + 1$, then define $g : V(G') \rightarrow \{0, 1, 2\}$ by $g(u_1) = g(u_2) = \ldots = g(u_k) = 2$, $g(v_1) = g(v_2) = \ldots = g(v_k) = 2$, and $g(u_i) = g(v_j) = 0$ for $k + 1 \leq i \leq p$ and $k \leq j \leq q$. Then
g is a Roman $k$-dominating function on $G'$ of weight $4k$ and thus $\gamma_kR(G') = \gamma_kR(G)$ and therefore $b_kR(G) \geq p - k + 1$.

If $H$ is the graph obtained from $G$ by removing the edge set $\{u_kv_k, u_{k+1}v_{k+1}, \ldots, u_pv_p\}$, then it is straightforward to verify that $\gamma_kR(H) \geq \min\{k + p + 1, 4k + 1\}$. Hence $b_kR(G) \leq p - k + 1$, and the proof is complete.

4. Conclusions

We introduced the Roman $k$-bondage number improving the concept of Roman bondage number and we characterized all graphs with the Roman $k$-bondage number equal to 1 and 2 and also determined the Roman $k$-bondage number of complete graphs. Also we present some upper bounds on Roman bondage number of general graphs.

It would be interesting to determine the Roman $k$-bondage number for some well known classes of graphs such as cubic graphs, chordal graphs, $n$-cubes, or Cartesian product of two graphs.

We conclude this section with four open problems.

**Problem 4.1.** Prove or disprove: For any connected graph $G$ of order $n \geq 3$, $b_R(G) = n - 1$ if and only if $G \cong K_3$.

As we have seen that $\gamma_R(C_4) = 3$ and $b_R(C_4) = 2$, so there exist graphs for which $b_R(G) = (\gamma_R(G) - 2)\Delta(G)$. We do not know whether the bound in Theorem 2.3 can be lowered by 1 and hence we pose:

**Problem 4.2.** If $G$ is a connected graph of order $n \geq 4$ with Roman domination number $\gamma_R(G) \geq 3$, then

$$b_R(G) \leq (\gamma_R(G) - 2)\Delta(G).$$

**Problem 4.3.** Prove or disprove: If $G$ is a connected graph of order $n \geq 3$, then

$$b_R(G) \leq n - \gamma_R(G) + 1.$$

**Problem 4.4.** Determine sharp (constant) upper bounds on $b_kR(G)$ for any graph $G$ with $\gamma_kR(G) < n$. In particular, for any graph $G$ of order $n$ with $\gamma_kR(G) < n$, is $b_kR(G) \leq n - 1$?

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References


