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On the Roman k-bondage number of a graph

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Abstract

A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex $v \in V$ with $f(v) = 0$ has at least one neighbor $u \in V$ with $f(u) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted by $\gamma_R(G)$. The Roman bondage number $b_R(G)$ of a graph G with maximum degree at least two is the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$. In this note we first present sharp bounds for $b_R(G)$ and then we initiate the study of the Roman k-bondage number in graphs. Some of our results extend those given by Jafari Rad and Volkmann in 2011 for the Roman bondage number.

Keywords: Roman domination number, Roman bondage number,

Roman k-domination number, Roman k-bondage number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *complement* $\overline{G}$ of G is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. The reader is referred to [6, 17] for any terminology and notation not defined here.

Let k be a positive integer. A subset S of vertices of G is a *k-dominating set* if $|N(v) \cap S| \geq k$ for every $v \in V - S$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. The k-bondage number of a graph G with $\Delta \geq k$ is the minimum cardinality among all sets of edges $B \subseteq E$ for which $\gamma_k(G - B) > \gamma_k(G)$. The k-bondage number was introduced by Lu and Xu [12].

Let $k \geq 1$ be an integer. Following Kämmerling and Volkmann [10], a *Roman k-dominating function* on a graph G is a labeling $f : V \to \{0, 1, 2\}$ such that every vertex with label 0 has at least k neighbors with label 2. The weight of a Roman k-dominating function is the value $f(V) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman k-dominating function on a graph G is called the *Roman k-domination number*, denoted by $\gamma_{kR}(G)$. Note that the Roman 1-domination number $\gamma_{1R}(G)$ is the usual Roman domination number $\gamma_R(G)$. A $\gamma_{kR}(G)$-*function* is a Roman k-dominating function on G with weight $\gamma_{kR}(G)$. A Roman k-dominating function $f : V \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a k-dominating set when f is an RkDF, and since placing weight 2 at the vertices of a k-dominating set yields an RkDF, in [10], it was observed that

$$\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G). \tag{1}$$

The definition of the Roman dominating function was given implicitly by Stewart [16] and ReVelle and Rosing [15]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [2] as well as Chambers, Kinnersley, Prince and West [1] have given a lot of results on Roman domination. For more information on Roman domination we refer the reader to [3, 4, 5, 7, 8, 9, 11].

Let k be a positive integer, and let G be a graph with maximum degree at least two. The *Roman k-bondage number* $b_{kR}(G)$ of G is the minimum cardinality of all sets $E' \subseteq E$ for which $\gamma_{kR}(G - E') > \gamma_{kR}(G)$. When $k = 1$, the Roman k-bondage number $b_{kR}(G)$ is the usual Roman bondage number $b_R(G)$ which was introduced by Jafari Rad and Volkmann in [13], and has been further studied for example in [14].

Our purpose in this paper is to initiate the study of the Roman k-bondage number in graphs. We first present some general upper bounds for Roman bondage number and then we study basic properties and bounds for the Roman k-bondage number of a graph. In addition, we determine the Roman k-bondage number of some classes of graphs.

We make use of the following observations and results for our investigations.

Observation 1.1. *Let k be a positive integer, and let G be a graph of order n with $\gamma_{kR}(G) = n$. Then for any $E' \subseteq E$ we have $\gamma_{kR}(G) = n = \gamma_{kR}(G - E')$.*

Observation 1.2. *Let k be a positive integer, and let G be a graph of order n. If $n \leq 2k$ or $\Delta < k$, then $\gamma_{kR}(G) = n$.*

Proposition A. [10] *If G is a graph of order n and maximum degree $\Delta = k$, then $\gamma_{kR}(G) = n$.*

Proposition B. [10] *If G is a graph of order n with at most one cycle, then $\gamma_{kR}(G) = n$ when $k \geq 2$.*

Proposition C. [10] *Let G be a graph of order n. Then $\gamma_{kR}(G) < n$ if and only if G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $d_H(v) \geq k$ for each $v \in X$.*

Proposition D. [13] *For $n \geq 3$, $b_R(K_n) = \lceil n/2 \rceil$.*

Proposition E. [13] *If G is a graph, and uvw a path of length 2 in G, then*

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 3.$$

If u and w are adjacent, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 4.$$

Regarding Observations 1.1, 1.2 and Propositions A and B, in the study of the Roman k-bondage $b_{kR}(G)$ we will assume that $\gamma_{kR}(G) < n$, $n > 2k$, $\Delta > k$ and that G is not a tree or unicyclic graph.

We start with the following observations and properties.

Observation 1.3. *Let G be a graph. Suppose q edges can be removed from G to give a graph H with $b_{kR}(H) = 1$. Then $b_{kR}(G) \leq q + 1$.*

Observation 1.4. *Let G be a graph of order n with $\gamma_{kR}(G) < n$. Assume that H is a spanning subgraph of G with $\gamma_{kR}(H) = \gamma_{kR}(G)$. If $K = E(G) - E(H)$, then $b_{kR}(H) \leq b_{kR}(G) \leq b_{kR}(H) + |K|$.*

If P_n and C_n are the path and cycle of order n, then it was shown in [2] that $\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil$. In addition, we find in [13] for $n \geq 3$:

1. $b_R(P_n) = 2$ if $n \equiv 2 \pmod 3$ and $b_R(P_n) = 1$ otherwise,
2. $b_R(C_n) = 3$ if $n \equiv 2 \pmod 3$ and $b_R(C_n) = 2$ otherwise.

Using these results and Observation 1.4, we obtain:

Corollary 1.5. *Let G be a graph of order $n \geq 3$.*

1. *If G has a Hamiltonian path and $\gamma_R(G) = \lceil 2n/3 \rceil$, then $b_R(G) \geq 2$ if $n \equiv 2 \pmod 3$,*
2. *If G is Hamiltonian with $\gamma_R(G) = \lceil 2n/3 \rceil$, then $b_R(G) \geq 2$ and $b_R(G) \geq 3$ if $n \equiv 2 \pmod 3$.*

Observation 1.6. *If a graph G has a vertex v such that $\gamma_{kR}(G-v) \geq \gamma_{kR}(G)$, then $b_{kR}(G) \leq \deg(v) \leq \Delta$.*

Observation 1.7. *If a graph G has a vertex v such that every $\gamma_{kR}(G)$-function assigns 2 to v, then $b_{kR}(G) \leq \deg(v) \leq \Delta$.*

Proof. If E_v is the set of edges incident with v, then we will show that $\gamma_{kR}(G-E_v) > \gamma_{kR}(G)$. Suppose to the contrary that $\gamma_{kR}(G-E_v) \leq \gamma_{kR}(G)$. Then $\gamma_{kR}(G-E_v) = \gamma_{kR}(G-v)+1$ and hence $\gamma_{kR}(G) \geq \gamma_{kR}(G-v)+1$. Since $\gamma_{kR}(G) \leq \gamma_{kR}(G-v)+1$, we obtain $\gamma_{kR}(G) = \gamma_{kR}(G-v)+1$, a contradiction to the hypothesis that every $\gamma_{kR}(G)$-function assigns 2 to v. □

2. Bounds on the Roman bondage number

In this section we establish bounds on the Roman bondage number of a graph that are independent of the graph structure.

Theorem 2.1. *If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq n-1.$$

Furthermore, this bound is sharp for K_3.

Proof. Let u and v be two adjacent vertices with $\deg(u) \leq \deg(v)$. If $b_R(G) \leq \deg(u) \leq n-1$, then we are done. Suppose to the contrary that $b_R(G) > \deg(u)$.

Let E_u denote the set of edges incident with u. Then $\gamma_R(G-E_u) = \gamma_R(G)$ and $\gamma_R(G-u) = \gamma_R(G)-1$. Assume that $\mathcal{F} = \{f = (V_0^f, V_1^f, V_2^f) \mid f \text{ is a } \gamma_R(G-u)-\text{function}\}$ and $V_2 = \cup_{f\in\mathcal{F}} V_2^f$. Then u is adjacent to no vertex of V_2 in G. Hence $|E_u| \leq n-1-|V_2|$ and $v \notin V_2$. Let F_v denote the set of edges from v to a vertex in V_2. If $\gamma_R(G-u-F_v) > \gamma_R(G-u)$ or equivalently $\gamma_R(G-u-F_v) > \gamma_R(G)-1$, then $\gamma_R(G-(E_u \cup F_v)) > \gamma_R(G)$, and we see that

$$b_R(G) \leq |E_u \cup F_v| \leq (n-1-|V_2|)+|V_2| = n-1.$$

So we assume that $\gamma_R(G-u-F_v) = \gamma_R(G-u)$. Then $\gamma_R(G-u-v) = \gamma_R(G-u)-1$. Since G is connected of order $n \geq 3$ and since $\deg(u) \leq \deg(v)$, we may assume that $w \in N(v)-\{u\}$. Let $\mathcal{F}' = \{f = (V_0^f, V_1^f, V_2^f) \mid f \text{ is a } \gamma_R(G-u-v)-\text{function}\}$, and let $V_2' = \cup_{f\in\mathcal{F}'} V_2^f$. Then u and v are not adjacent to any vertex of V_2' in G. Let F_w denote the set of edges from w to a vertex in V_2' and so $w \notin V_2'$.

If $w \in V_1^h$ for some $h \in \mathcal{F}'$, then $\gamma_R(G-\{u,v,w\}) = \gamma_R(G-\{u,v\})-1 = \gamma_R(G)-3$ and obviously the function $g: V(G) \to \{0,1,2\}$ defined by

$$g(u) = g(w) = 0, g(v) = 2 \text{ and } g(x) = h(x) \text{ for } x \in V(G)-\{u,v,w\},$$

is a Roman dominating function on G of weight less than $\gamma_R(G)$ which is a contradiction. Therefore $\gamma_R(G-u-v-F_w) > \gamma_R(G-u-v)$ or equivalently $\gamma_R(G-u-v-F_w) > \gamma_R(G)-2$. Thus $\gamma_R(G-(E_u \cup F_v \cup F_w)) > \gamma_R(G)$ and we see that

$$b_R(G) \le |E_u \cup F_v \cup F_w| \le (n-1-|V_2'|) + |V_2'| = n-1,$$

and the proof is complete. □

By a closer look at the proof of Theorem 2.1 we have the following result improving Proposition E.

Corollary 2.2. *If G is a connected graph of order $n \ge 3$ and uvw a path of length 2 in G, then*

$$b_R(G) \le \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - |N(w) \cap (N(u) \cup N(v)) - \{u,v\}| - 3.$$

If u and w are adjacent, then

$$b_R(G) \le \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - |N(w) \cap (N(u) \cup N(v)) - \{u,v\}| - 4.$$

The next result presents an upper bound on the Roman bondage number that involves the maximum degree. This bound also indicates a relationship between the Roman bondage number and the Roman domination number. If $\gamma_R(G) = 2$, then obviously $b_R(G) \le \delta(G)$. So we assume that $\gamma_R(G) \ge 3$.

Theorem 2.3. *If G is a connected graph of order $n \ge 4$ with Roman domination number $\gamma_R(G) \ge 3$, then*

$$b_R(G) \le (\gamma_R(G) - 2)\Delta(G) + 1.$$

Proof. The proof is by induction on $\gamma_R(G)$. First suppose $\gamma_R(G) = 3$. Let u be a vertex of maximum degree in G and let E_u denote the set of edges incident with u. If $\gamma_R(G - E_u) > \gamma_R(G)$, then $b_R(G) \le |E_u| = \deg(u)$ and hence $b_R(G) \le \Delta(G)$. Assume that $\gamma_R(G - E_u) = \gamma_R(G)$ or equivalently $\gamma_R(G - u) = \gamma_R(G) - 1 = 2$. Since $n \ge 3$, there is a vertex v that is adjacent with every vertex of G but u. Thus $\deg_G(v) = \Delta$ also, and u is adjacent with every vertex of G except v. If there is an edge e incident with v such that $\gamma_R(G - u - e) > \gamma_R(G - u)$, then obviously $b_R(G) \le \deg(u) + 1 \le \Delta + 1$ and we are done. Assume that the removal from $G - u$ of any one edge incident with v again leaves a graph with Roman domination number 2. It follows that there is a vertex $w \ne v$ that is adjacent to every vertex of $G - u$. Since the vertex v is the only vertex of G that is not adjacent with u, we deduce that w must be adjacent in G with u. This however implies that $\gamma_R(G) = 2$ which is a contradiction.

Now assume that the statement is true for any graph of order $n \ge 4$ with Roman domination number $3 \le \gamma_R(G) \le k$. Let G be a graph of order $n \ge 4$ with $\gamma_R(G) = k+1$. Suppose to the contrary that $b_R(G) > (\gamma_R(G) - 2)\Delta(G) + 1$. Then for any vertex u of

G, we have $\gamma_R(G-u)=\gamma_R(G)-1$, since $\deg(u)<b_R(G)$. By Observation 1.4 we have $b_R(G)\le b_R(G-u)+\deg(u)$, and by the induction hypothesis we have

$$b_R(G)\le (k-2)\Delta(G-u)+1+\deg(u)=(k-1)\Delta(G)+1,$$

as desired. This completes the proof. □

3. Properties of the Roman k-bondage number

In this section we first characterize the graphs whose Roman k-bondage number is one or two and then we determine the Roman k-bondage number of some classes of graphs.

Let G be a graph and D be a subset of $V(G)$. For any $x\in D$, a vertex y not in D is said to be a k-private neighbor of x with respect to D if y is a neighbor of x and $|D\cap N_G(y)|=k$. The k-private neighborhood of x with respect to D, denoted by $PN_k(x,D,G)$, is the set of all k-private neighbors of x with respect to D in G.

Theorem 3.1. *Let k be a positive integer and let G be a graph with $\Delta(G)\ge k+1$. Then $b_{kR}(G)=1$ if and only if G has an edge xy satisfying either $x\in V_2$ and $y\in V_0\cap PN_k(x,V_2,G)$ or $y\in V_2$ and $x\in V_0\cap PN_k(y,V_2,G)$ for any $\gamma_{kR}(G)$-function $f=(V_0,V_1,V_2)$.*

Proof. Assume that $b_{kR}(G)=1$. Then there is an edge, denoted by xy, in G such that $\gamma_{kR}(G-xy)>\gamma_{kR}(G)$. Let $f=(V_0,V_1,V_2)$ be a $\gamma_{kR}(G)$-function. If $x\in V_1$ or $y\in V_1$, then obviously f is a $\gamma_{kR}(G-xy)$-function which is a contradiction. If $x,y\in V_0$ or $x,y\in V_2$, then clearly f is a $\gamma_{kR}(G-xy)$-function which is a contradiction again. Thus we may assume, without loss of generality, that $x\in V_2$ and $y\in V_0$. Then y has at least k neighbors in V_2. If $|V_2\cap N_G(y)|>k$ then f is a $\gamma_{kR}(G-xy)$-function, a contradiction. Thus $|V_2\cap N_G(y)|=k$ and hence $y\in V_0\cap PN_k(x,V_2,G)$.

Conversely, we assume that the edge xy of G satisfies either $x\in V_2$ and $y\in V_0\cap PN_k(x,V_2,G)$ or $y\in V_2$ and $x\in V_0\cap PN_k(y,V_2,G)$ for any $\gamma_{kR}(G)$-function $f=(V_0,V_1,V_2)$. Then we only need to prove $\gamma_{kR}(G-xy)>\gamma_{kR}(G)$. Suppose to the contrary that $\gamma_{kR}(G-xy)=\gamma_{kR}(G)$.

Let $g=(V_0^g,V_1^g,V_2^g)$ be a $\gamma_{kR}(G-xy)$-set. Since $\gamma_{kR}(G-xy)=\gamma_{kR}(G)$, g is a $\gamma_{kR}(G)$-set. Then we may assume, without loss of generality, that $x\in V_2^g$ and $y\in V_0^g\cap PN_k(x,V_2^g,G)$. This implies that $|V_2^g\cap N_{G-xy}(y)|=k-1$ which is a contradiction. This completes the proof. □

Corollary 3.2. *Let k be a positive integer, and let G be a graph of order n with $\gamma_{kR}(G)<n$. If G has a unique $\gamma_{kR}(G)$-function, then $b_{kR}(G)=1$.*

Proof. Let $f=(V_0^f,V_1^f,V_2^f)$ be the $\gamma_{kR}(G)$-function. If there exists a vertex $u\in V_0^f$ with $v\in V_2^f\cap N(u)$ such that $u\in PN_k(v,V_2,G)$, then the result follows from Theorem 3.1. Let

$|N(u) \cap V_2^f| \geq k+1$ for each vertex $u \in V_0^f$. Then for each $v \in V_2^f$, $(V_0^f, V_1^f \cup \{v\}, V_2^f - \{v\})$ is a RkDF of G with weight less than $\gamma_{kR}(G)$ which is a contradiction. This completes the proof. □

Theorem 3.3. *Let k be a positive integer, and let G be a graph with $\Delta(G) \geq k+1$. Then $b_{kR}(G) = 2$ if and only if G has two edges u_1v_1, u_2v_2 satisfying one of the following conditions:*

1. *$u_1 = u_2, u_1 \in V_2$ and $v_1 \in V_0 \cap PN_k(u_1, V_2, G)$ or $v_2 \in V_0 \cap PN_k(u_1, V_2, G)$ for any $\gamma_{kR}(G)$-function $f = (V_0, V_1, V_2)$ and there exists $\gamma_{kR}(G)$-functions f_i $(i = 1, 2)$ such that $v_i \in V_0 \cap PN_k(u_1, V_2^{f_i}, G)$ for each i.*
2. *$u_1 = u_2 \in V_0, v_1, v_2 \in V_2$ and $u_1 \in V_0 \cap PN_{k+1}(v_1, V_2, G) \cap PN_{k+1}(v_2, V_2, G)$,*
3. *$u_1 = u_2 \in V_0, v_1 \in V_2$ and $u_1 \in V_0 \cap PN_k(v_1, V_2, G)$ or $v_2 \in V_2$ and $u_1 \in V_0 \cap PN_k(v_2, V_2, G)$ and there exists $\gamma_{kR}(G)$-functions f_i $(i = 1, 2)$ such that $v_i \in V_2^{f_i}$ and $u_1 \in V_0 \cap PN_k(v_i, V_2^{f_i}, G)$ for each i,*
4. *u_1v_1, u_2v_2 are independent, $\{u_1, v_1\} \cap V_2 \neq \emptyset$, say $v_1 \in V_2$, and $u_1 \in V_0 \cap PN_k(v_1, V_2, G)$ or $\{u_1, v_1\} \cap V_2 \neq \emptyset$, say $v_2 \in V_2$, and $u_2 \in V_0 \cap PN_k(v_1, V_2, G)$, and there exists $\gamma_{kR}(G)$-functions f_i $(i = 1, 2)$ such that $u_i \in V_0 \cap PN_k(v_i, V_2^{f_i}, G)$ for each i.*

for any $\gamma_{kR}(G)$-function $f = (V_0, V_1, V_2)$.

Proof. Assume that $b_{kR}(G) = 2$. Then there are two edges, denoted by u_1v_1, u_2v_2, in G such that $\gamma_{kR}(G - \{u_1v_1, u_2v_2\}) > \gamma_{kR}(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G)$-function. We consider two cases.

Case 1. $\{u_1, v_1\} \cap \{u_2, v_2\} \neq \emptyset$.

Then we may assume, without loss of generality, that $u_1 = u_2$. If $u_1 \in V_1$, then obviously f is a $\gamma_{kR}(G - \{u_1v_1, u_2v_2\})$-dominating function which is a contradiction. Let $u \in V_2$. If $v_1 \notin V_0 \cap PN_{k+1}(u_1, V_2, G)$ and $v_2 \notin V_0 \cap PN_{k+1}(u_1, V_2, G)$, then f is a $\gamma_{kR}(G - \{u_1v_1, u_2v_2\})$-dominating function, a contradiction. Thus $v_1 \in V_0 \cap PN_{k+1}(u_1, V_2, G)$ or $v_2 \in V_0 \cap PN_{k+1}(u_1, V_2, G)$. Since $b_{kr}(G) = 2$, it follows from Theorem 3.1 that there exist $\gamma_{kR}(G)$-functions f_i $(i = 1, 2)$ such that $v_i \in V_0 \cap PN_{k+1}(u_1, V_2^{f_i}, G)$ for each i. Therefore the condition 1 is fulfilled.

Finally, let $u_1 \in V_0$. If $v_1, v_2 \in V_2$ and $u_1 \in PN_{k+1}(v_1, V_2, G)$ and $u_1 \in PN_{k+1}(v_2, V_2, G)$, then the condition 2 is fulfilled. Let $\{v_1, v_2\} \nsubseteq V_2$, $u_1 \notin PN_{k+1}(v_1, V_2, G)$ or $u_1 \notin PN_{k+1}(v_2, V_2, G)$. If $v_1, v_2 \notin V_2$, $v_1 \in V_2$ and $u_1 \notin PN_k(v_1, V_2, G)$ or $v_2 \in V_2$ and $u_1 \notin PN_k(v_2, V_2, G)$, then obviously f is a $\gamma_{kR}(G - \{u_1v_1, u_2v_2\})$-dominating function, a contradiction. Thus we may assume $v_1 \in V_2$ and $u_1 \in PN_k(v_1, V_2, G)$ or $v_2 \in V_2$ and $u_1 \in PN_k(v_1, V_2, G)$. Since $b_{kr}(G) = 2$, it follows from Theorem 3.1 that there exist

$\gamma_{kR}(G)$-functions f_i $(i = 1, 2)$ such that $v_i \in V_2^{f_i}$ and $u_1 \in PN_k(v_i, V_2^{f_i}, G)$. Hence the condition 3 holds.

Case 2. $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$.

Then the edges u_1v_1 and u_2v_2 are independent. If $\{u_i, v_i\} \cap V_2 = \emptyset$ for $i = 1, 2$, then obviously f is a $\gamma_{kR}(G - \{u_1v_1, u_2v_2\})$-function which is a contradiction. Thus $\{u_i, v_i\} \cap V_2 \neq \emptyset$ for some i. Let $\{u_1, v_1\} \cap V_2 \neq \emptyset$ (the proof of the case $\{u_2, v_2\} \cap V_2 \neq \emptyset$ is similar). If $\{u_1, v_1\} \subseteq V_2$, then f is a $\gamma_{kR}(G - \{u_1v_1\})$-function. If $\{u_2, v_2\} \cap V_1 \neq \emptyset$, $\{u_2, v_2\} \subseteq V_0$ or $\{u_2, v_2\} \subseteq V_2$, then f is a $\gamma_{kR}(G - \{u_1v_1, u_2v_2\})$-function which is a contradiction. Therefore we may assume, without loss of generality, that $v_2 \in V_2$ and $u_2 \in V_0$. Since $\gamma_{kR}(G - \{u_1v_1, u_2v_2\}) > \gamma_{kR}(G - \{u_1v_1\}) = \gamma_{kR}(G)$, we must have $u_2 \in PN_k(v_2, V_2, G)$.

If $v_2 \in V_2$ and $u_2 \in PN_k(v_2, V_2, G)$ or $u_2 \in V_2$ and $v_2 \in PN_k(u_2, V_2, G)$ for every $\gamma_{kR}(G)$-function, then $b_{kR}(G) = 1$ by Theorem 3.1, a contradiction. Thus there is a $\gamma_{kR}(G)$-function, say g, such that $v_2 \notin V_2$ or $u_2 \notin PN_k(v_2, V_2, G)$ and $u_2 \notin V_2$ or $v_2 \notin PN_k(u_2, V_2, G)$. Then g is a $\gamma_{kR}(G - u_2v_2)$-function and hence $\gamma_{kR}(G) = \gamma_{kR}(G - u_2v_2)$. An argument similar to that described above $v_1 \in V_2$ and $u_1 \in PN_k(v_1, V_2, G)$ or $u_1 \in V_2$ and $v_1 \in PN_k(u_1, V_2, G)$. Thus the condition 3 holds.

Conversely, we assume that G has two edges u_1v_1, u_2v_2 satisfying one of the above conditions. It is easy to see that $\gamma_{kR}(G - \{u_1v_1, u_2v_2\}) > \gamma_{kR}(G)$. This completes the proof. □

Theorem 3.4. *Let k be a positive integer, and let G be a graph of order $n > 2k$. If G has exactly $m \geq k$ vertices of degree $n - 1$, then*

$$b_{kR}(G) = \left\lceil \frac{m - k + 1}{2} \right\rceil.$$

Proof. Since $n > 2k$ and $m \geq k$, $\gamma_{kR}(G) = 2k$. Suppose that $E' \subseteq E(G)$ is an arbitrary subset of edges such that $|E'| < \lceil \frac{m-k+1}{2} \rceil$. It is clear that $G - E'$ has at least k vertices of degree $n - 1$ and hence $\gamma_{kR}(G - E') = 2k$. It follows that $b_{kR}(G) \geq \lceil \frac{m-k+1}{2} \rceil$.

Let $S = \{v_1, v_2, \ldots, v_m\}$ be the set of vertices of degree $n - 1$ in G. Then the subgraph $G[S]$ is the complete graph K_m. If $m - k + 1$ is even, then let H_1 be the graph obtained from G by removing $\frac{m-k+1}{2}$ independent edges from $G[S]$. Then H_1 has exactly $k - 1$ vertices of degree $n - 1$. This implies that $\gamma_{kR}(H_1) \geq 2k + 1$. Hence $b_{kR}(G) \leq \lceil \frac{m-k+1}{2} \rceil$.

If $m - k + 1$ is odd, then let H_2 be the graph obtained from G by removing $\frac{m-k+2}{2}$ independent edges from $G[S]$. Then H_2 has exactly $k - 2$ vertices of degree $n - 1$. Thus $\gamma_{kR}(H_2) \geq 2k + 1$. This implies that $b_{kR}(G) \leq \lceil \frac{m-k+1}{2} \rceil$. Combining the obtained inequalities, we have $b_{kR}(G) = \lceil \frac{m-k+1}{2} \rceil$, and the proof is complete. □

The special case $k = 1$ of the next result can be found in [13].

Corollary 3.5. *If $n > 2k$, then $b_{kR}(K_n) = \lceil \frac{n-k+1}{2} \rceil$.*

If G is isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p$, then by Proposition C, $\gamma_{kR}(G) = p + q = n$ when $p < k$ or $q = p = k$. Now we determine the Roman k-domination number for complete bipartite graphs $K_{p,q}$ with $q \geq p \geq k$ and $p+q \geq 2k+1$.

Theorem 3.6. *Let k be a positive integer, and let G be isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p \geq k$ such that $n = p + q \geq 2k + 1$. Then*

$$\gamma_{kR}(G) = \min\{k + p, 4k\}.$$

Proof. Let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$ be the partite sets of G.

If $p \leq 3k$, then define $f : V(G) \longrightarrow \{0, 1, 2\}$ by $f(u_1) = f(u_2) = \ldots = f(u_k) = 2$, $f(u_{k+1}) = f(u_{k+2}) = \ldots = f(u_p) = 1$ and $f(v_i) = 0$ for $1 \leq i \leq q$. Then f is a Roman k-dominating function of weight $2k + p - k = k + p$ and thus $\gamma_{kR}(G) \leq k + p < p + q = n$.

If $p \geq 3k + 1$, then define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(u_1) = g(u_2) = \ldots = g(u_k) = 2$, $g(v_1) = g(v_2) = \ldots = g(v_k) = 2$, and $g(u_i) = g(v_j) = 0$ for $k \leq i \leq p$ and $k \leq j \leq q$. Then g is a Roman k-dominating function of weight $4k$ and thus $\gamma_{kR}(G) \leq 4k < p + q = n$.

Now let $f = (V_0, V_1, V_2)$ be any $\gamma_{kR}(G)$-function. If $V_0 = \emptyset$, then $\gamma_{kR}(G) = n$, a contradiction. Hence we can assume that $|V_0| \geq 1$. If $V_0 \cap X \neq \emptyset$ and $V_0 \cap Y \neq \emptyset$, then $|V_2 \cap Y| \geq k$ and $|V_2 \cap X| \geq k$ and thus $\gamma_{kR}(G) \geq 4k$. If $V_0 \cap X \neq \emptyset$ and $V_0 \cap Y = \emptyset$, then $|V_2 \cap Y| = k$ and $|V_1 \cap Y| = q - k$ and thus $\gamma_{kR}(G) \geq 2k + q - k = k + q$. If $V_0 \cap X = \emptyset$ and $V_0 \cap Y \neq \emptyset$, then $|V_2 \cap X| = k$ and $|V_1 \cap X| = p - k$ and thus $\gamma_{kR}(G) \geq 2k + p - k = k + p$. Combining the last three inequalities, we see that

$$\gamma_{kR}(G) \geq \min\{k + p, k + q, 4k\} = \min\{k + p, 4k\},$$

and altogether, we obtain the desired result. □

Theorem 3.7. *Let k be a positive integer, and let G be isomorphic to the complete bipartite graph $K_{p,q}$ with $q \geq p \geq k$ such that $n = p + q \geq 2k + 1$. Then*

$$b_{kR}(G) = p - k + 1.$$

Proof. Let $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$ be the partite sets of G. Suppose that $E' \subseteq E(G)$ is an arbitrary edge set with $|E'| \leq p - k$, and let $G' = G - E'$. Then $X \cap V(G')$ contains at least k vertices, say $u_1, u_2, \ldots, u_k$ of degree q and $Y \cap V(G')$ contains at least k vertices, say $v_1, v_2, \ldots, v_k$ of degree p.

If $p \leq 3k$, then define $f : V(G') \longrightarrow \{0, 1, 2\}$ by $f(u_1) = f(u_2) = \ldots = f(u_k) = 2$, $f(u_{k+1}) = f(u_{k+2}) = \ldots = f(u_p) = 1$ and $f(v_i) = 0$ for $1 \leq i \leq q$. Then f is a Roman k-dominating function on G' of weight $2k + p - k = k + p$ and thus $\gamma_{kR}(G') \leq k + p$. It follows from Theorem 3.6 that $\gamma_{kR}(G') = \gamma_{kR}(G)$ and therefore $b_{kR}(G) \geq p - k + 1$.

If $p \geq 3k + 1$, then define $g : V(G') \longrightarrow \{0, 1, 2\}$ by $g(u_1) = g(u_2) = \ldots = g(u_k) = 2$, $g(v_1) = g(v_2) = \ldots = g(v_k) = 2$, and $g(u_i) = g(v_j) = 0$ for $k \leq i \leq p$ and $k \leq j \leq q$. Then

g is a Roman k-dominating function on G' of weight $4k$ and thus $\gamma_{kR}(G) \leq 4k$. It follows from Theorem 3.6 that $\gamma_{kR}(G') = \gamma_{kR}(G)$ and therefore $b_{kR}(G) \geq p - k + 1$.

If H is the graph obtained from G by removing the edge set $\{u_k v_k, u_{k+1} v_{k+1}, \ldots, u_p v_p\}$, then it is straightforward to verify that $\gamma_{kR}(H) \geq \min\{k+p+1, 4k+1\}$. Hence $b_{kR}(G) \leq p - k + 1$, and the proof is complete. $\square$

4. Conclusions

We introduced the Roman k-bondage number improving the concept of Roman bondage number and we characterized all graphs with the Roman k-bondage number equal to 1 and 2 and also determined the Roman k-bondage number of complete graphs. Also we present some upper bounds on Roman bondage number of general graphs.

It would be interesting to determine the Roman k-bondage number for some well known classes of graphs such as cubic graphs, chordal graphs, n-cubes, or Cartesian product of two graphs.

We conclude this section with four open problems.

Problem 4.1. *Prove or disprove: For any connected graph G of order $n \geq 3$, $b_R(G) = n-1$ if and only if $G \simeq K_3$.*

As we have seen that $\gamma_R(C_4) = 3$ and $b_R(C_4) = 2$, so there exist graphs for which $b_R(G) = (\gamma_R(G) - 2)\Delta(G)$. We do not know whether the bound in Theorem 2.3 can be lowered by 1 and hence we pose:

Problem 4.2. *If G is a connected graph of order $n \geq 4$ with Roman domination number $\gamma_R(G) \geq 3$, then*

$$b_R(G) \leq (\gamma_R(G) - 2)\Delta(G).$$

Problem 4.3. *Prove or disprove: If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq n - \gamma_R(G) + 1.$$

Problem 4.4. *Determine sharp (constant) upper bounds on $b_{kR}(G)$ for any graph G with $\gamma_{kR}(G) < n$. In particular, for any graph G of order n with $\gamma_{kR}(G) < n$, is $b_{kR}(G) \leq n-1$?*

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