

k -KERNELS IN MULTIPARTITE TOURNAMENTS

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Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D , respectively.

A (k, l) -kernel N of D is a k -independent set of vertices (if $u, v \in N$ then $d(u, v), d(v, u) \geq k$) and l -absorbent (if $u \in V(D) - N$ then there exists $v \in N$ such that $d(u, v) \leq l$). A k -kernel is a $(k, k - 1)$ -kernel. An m -partite tournament is an orientation of an m -partite complete graph. In this paper we use a tool for finding sufficient conditions for a digraph to have a k -kernel, the k -closure of a digraph, which is employed to prove that every m -partite tournament has a k -kernel for every $k \geq 4$, $m \geq 2$, and to give a simple proof of the fact that for $k \geq 2$ it is NP -complete to decide if a digraph D has a k -kernel, or not. We also characterize the m -partite tournaments that have a 3-kernel for $m \geq 2$ as those m -partite tournaments having a 2-absorbent vertex for the directed cycles of length four.

Keywords: digraph, kernel, (k, l) -kernel, k -kernel, tournament, multipartite tournament.

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1. Introduction

For general concepts and notation we refer the reader to [1, 2, 15], particularly we will use the notation of [15] for walks, if $\mathcal{C} = (x_0, x_1, \dots, x_n)$ is a walk and $i < j$, then $x_i \mathcal{C} x_j$ will denote the subwalk of \mathcal{C} , $(x_i, x_{i+1}, \dots, x_{j-1}, x_j)$. Union of walks will be denoted by concatenation or with \cup . An arc $(u, v) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). Given a digraph D , a subset S of its vertices is called *absorbing* if for every vertex $u \in V(D) \setminus S$ there exists a vertex $v \in S$ such that $(u, v) \in A(D)$.

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G , where $\chi(G)$ and $\omega(G)$ denote the chromatic and clique number of G respectively. A graph G is called *Berge* if it does not contain an induced odd hole (chordless odd cycle of length ≥ 5) nor odd-antihole (complement of an odd hole). Claude Berge conjectured [3, 4] that a

graph is perfect if, and only if, it is Berge, this result, known as The Strong Perfect Graph Theorem, was proved by Chudnovsky, et al. [12].

Given a graph $G = (V(G), E(G))$, a biorientation of G is a digraph obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc (x, y) or (y, x) or both. A biorientation is called *clique-acyclic* if no clique contains a directed cycle.

The concept of *kernel* of a digraph was introduced in [41] by Von Neumann and Morgenstern in the context of game theory as an independent and absorbing set. Since then, this concept has been relevant in graph theory for its relations with a variety of problems, *e.g.*, in list colourings and perfectness; it has also found several applications in game theory. Berge was one of the pioneers in this area, finding sufficient and necessary conditions for the existence of kernels in digraphs (although no characterization is yet known), studying classes of digraphs that have kernel and successfully using kernels to solve problems in other areas of mathematics [5, 6, 7, 9].

Along with Duchet [8], Berge defined a graph to be *kernel solvable* if every clique-acyclic biorientation of it has a kernel. Maybe the most important result concerning kernels was conjectured by Berge and Duchet: a graph is perfect if, and only if, it is kernel-solvable. The fact that kernel solvable graphs are perfect follows from the Strong Perfect Graph Theorem and the converse was proved by Boros and Gurvich [11]. A very complete survey about kernels is the one of Boros and Gurvich [10]. Also the work of Galeana-Sánchez is relevant for the subject [17, 18, 19, 20, 21, 24, 25, 26].

Many concepts have arisen generalizing kernels, but a very natural one is that of (k, l) -kernel introduced by M. Kwaśnik in [31]. It is clear that a $(2, 1)$ -kernel is a kernel in the usual sense. As a special case of (k, l) -kernels we consider the k -kernels; we define a k -kernel to be a $(k, k - 1)$ -kernel. Under this definition, a kernel is a 2-kernel. Another special case of (k, l) -kernels is the $(2, 2)$ -kernels, also known as *quasi-kernels*. Chvátal and Lovász proved in [14] that every digraph has a $(2, 2)$ -kernel, and as immediate corollary we have a classical result due to Landau [34], affirming that every tournament has a 3-kernel and thus a k -kernel for every $k \geq 3$. The principal motivation for the present work was to generalize this result concerning tournaments to multipartite tournaments.

Recently some work has been done for k -kernels and (k, l) -kernels concerning large families of digraphs, like the study of products of digraphs and how the k -kernels are preserved (Włoch and Włoch [42], in particular with Szumny in [37, 38]), also special families like quasi-transitive or pre-transitive digraphs [22] or unilateral cyclically k -partite digraphs [23] have been studied by Galeana-Sánchez and Hernández-Cruz. Other classical results are those of Kwaśnik on the superdigraphs or certain families of digraphs (with Kucharska [30]) or the cyclically k -partite strong digraphs [32], and those of Galeana-Sánchez generalizing some results from kernels to k -kernels [16]. Despite this fact, the results on the field are still very restricted.

On the other hand, multipartite tournaments are among the most widely studied families of digraphs, as the survey of Volkmann [40] shows. This family has been studied in diverse contexts, such as hamiltonicity, pancyclicity, properties of cycles and paths, etc.

But a subject that has received a lot of attention is the existence and number of 3-kings and 4-kings. A vertex v in a digraph D is a k -king for D if $d(v, u) \leq k$ for every $u \in V(D)$ and is a k -serf for D if $d(u, v) \leq k$ for every $v \in V(D)$.

We call a vertex v in a digraph a *transmitter* if it has $d^-(v) = 0$ and a *receiver* if it has $d^+(v) = 0$. An obvious necessary condition for a digraph to have a k -king (k -serf) is that it has at most one transmitter (receiver). Clearly that restriction is not necessary for the case of the k -solutions (k -kernels). Gutin [27] and independently Petrovic and Thomassen [35] proved that every multipartite tournament with at most one transmitter has a 4-king, and there are infinitely many examples of multipartite tournaments without 3-kings, so the two directions that have been studied since then are to find the number and distribution of 4-kings in multipartite tournaments without a 3-king and sufficient conditions (or characterizations) for a multipartite tournament to have 3-kings [28, 29, 36, 39].

It is clear that a k -serf (k -king) is also a k -kernel (k -solution), but the converse is not true, so although sufficient conditions for the existence of k -kings can be transformed in sufficient conditions for the existence of k -serfs and thus for the existence of k -kernels, characterization theorems cannot be extended in neither way. As we have seen, k -kernels and k -solutions are generalizations of kernels and solutions, but they are also generalizations of k -serfs and k -kings, so a characterization theorem for multipartite tournaments having a 3-kernel is very valuable. In this work we give a theorem with two distinct such characterizations.

2. Dualization

The solution of a digraph is the dual notion of a kernel, so Kwaśnik's concept can be adopted to generalize the concept of solution to (k, l) -solution of a digraph.

Definition 2.1. *Let D be a digraph and $S \subseteq V(D)$.*

- *The set S is l -dominating if for every $v \in V(D) \setminus S$ there exists $u \in S$ such that $d(u, v) \leq l$.*
- *The set S is called a (k, l) -solution of D if it is both k -independent and l -dominating.*
- *A k -solution is a $(k, k - 1)$ -solution.*

It is straightforward to prove that if a digraph D has a k -kernel, then the digraph obtained by the reversing of all the arcs in D has a k -solution, so let us recall the formal definition of the dual of a digraph to state some results concerning this operation.

Definition 2.2. *If D is a digraph, the dual (or converse, or transpose) of D , \overleftarrow{D} is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u, v) \in A(\overleftarrow{D})$ if, and only if, $(v, u) \in A(D)$.*

Remark 2.3. Clearly $\overleftarrow{\overleftarrow{D}} = D$. Also it is obvious that if N is a (k, l) -kernel of D then N is a (k, l) -solution of \overleftarrow{D} .

Another trivial remark, which be useful for the aims of this work is the next.

Remark 2.4. If T is an m -partite tournament, then \overleftarrow{T} is also an m -partite tournament.

Is clear that the concept of k -serf is the dual concept of k -king and results about k -kings can be dualized to results about k -serfs in the same way results about k -kernels can be dualized to results about k -solutions, and in particular in multipartite tournaments, Remark 2.4 help us to dualize almost every such result in an efficient way.

3. k -closure

The idea of the k -closure is, given a digraph D , to build a new digraph $\mathcal{C}_k(D)$ such that $(k + 1)$ -independence in D is related with independence in $\mathcal{C}_k(D)$ as well as k -absorbency in D is related with absorbency in $\mathcal{C}_k(D)$, in such a way that we can reduce the problem of finding a k -kernel in D to find a kernel in $\mathcal{C}_k(D)$, and of course, use the known sufficient conditions for the existence of kernels in digraphs to derive existence of k -kernels in families of digraphs or sufficient conditions for the existence of k -kernels in families of digraphs. The k -closure of a digraph in the context of (k, l) -kernels was introduced by Kwaśnik, Włoch and Włoch in [33].

Definition 3.1. If D is a digraph and $k \in \mathbb{N}$, the k -closure of D is the digraph $\mathcal{C}_k(D)$ such that $V(\mathcal{C}_k(D)) = V(D)$ and $A(\mathcal{C}_k(D)) = \{(u, v) \mid \text{there is a } uv\text{-directed walk of length } \leq k \text{ in } D\}$.

We clearly have from the definition that $\mathcal{C}_1(D) = D$. The following lemma can be found in [33]. We include the proof because is an excellent example of how this tool can be used to deal with k -independence or k -domination problems.

Lemma 3.2. If D is a digraph then a set $N \subseteq V(D)$ is a kernel of $\mathcal{C}_k(D)$ if, and only if, N is a $(k + 1)$ -kernel of D .

Proof. Let N be a kernel of $\mathcal{C}_k(D)$. From the definition of $\mathcal{C}_k(D)$, if $x, y \in V(D)$ we have that $(x, y) \notin A(\mathcal{C}_k)$ if, and only if, it does not exist an xy -directed walk of length less than or equal to k in D . Since N is independent in $\mathcal{C}_k(D)$, if $x, y \in N$ then $(x, y), (y, x) \notin A(\mathcal{C}_k(D))$ and thus no xy -directed walk nor yx -directed walk of length less than or equal to k exist in D , therefore N is a $(k + 1)$ -independent set in D . Besides, as N is an absorbent set in $\mathcal{C}_k(D)$, for every $u \in V(D) \setminus N$ there exists a vertex $v \in N$ such that $(u, v) \in A(\mathcal{C}_k(D))$ and, in consequence, there exists a uv -directed walk of length less than or equal to k in D , thence N is a k -absorbent set in D ; then N is the $(k + 1)$ -kernel we have been looking for. Conversely, if N is a $(k + 1)$ -kernel of D , it follows from the $(k + 1)$ -independence

that for every $x, y \in N$ there is no xy -directed walk nor yx -directed walk of length less than or equal to k in D , and hence $(x, y), (y, x) \notin A(\mathcal{C}_k(D))$, so N is an independent set in $\mathcal{C}_k(D)$. If $u \in V(D) \setminus N$ we have from the k -absorbency of N that there exists $v \in N$ such that $d_D(u, v) \leq k$, it follows that $(u, v) \in A(\mathcal{C}_k(D))$ and hence N is an absorbent set in $\mathcal{C}_k(D)$. \square

Lemma 3.2 can be combined with all the known results about sufficient conditions for the existence of kernels in digraphs in a way that, if we can asseverate the existence of a kernel in $\mathcal{C}_k(D)$, then we will have as an immediate consequence the existence of a $(k + 1)$ -kernel in D .

A well known result we will use within this article is the next theorem due to Berge and Duchet [7].

Definition 3.3. *A digraph D is called kernel-perfect if every induced subdigraph of D has a kernel.*

Theorem 3.4. *If every directed cycle of D has at least one symmetrical arc, then D is kernel-perfect.*

There are other results imposing conditions on the directed cycles of a digraph as a sufficient condition for the existence of a kernel, so directed cycles in the k -closure of a digraph are an important structure to be considered. We may ask under what conditions the k -closure of a digraph does not have directed cycles of odd length, or every directed cycle has at least one symmetrical arc, or every directed cycle has at least two pseudo-diagonals, or whatever fits for the k -closure to have a kernel. As the first example using this technique, in the next section we present the forementioned result for multipartite tournaments.

Another result that has an interesting corollary is due to Chvátal [13].

Theorem 3.5. *It is NP-complete to recognize whether a directed graph has a kernel, or not.*

Corollary 3.6. *For every $k \geq 2$ it is NP-complete to recognize whether a directed graph has a k -kernel, or not.*

Proof. It is an immediate consequence of Theorem 3.5 and Lemma 3.2. \square

So, as anyone can imagine, finding a k -kernel in a digraph or recognizing if there is none is a problem as difficult as the analogous for kernels.

4. k -kernels in multipartite tournaments

Definition 4.1. *A multipartite tournament is an orientation of a complete multipartite graph. If T is a multipartite tournament, it is usually denoted by $T = (V_1, V_2, \dots, V_m)$, where $\{V_1, V_2, \dots, V_m\}$ is the vertex partition of T ; the sets V_i $i \in \{1, \dots, m\}$ are called the classes of T . If T has m classes then it is called an m -partite tournament.*

It has been already proved that every m -partite tournament has a 4-king [27, 35], and thus, also a 4-serf. Since a 4-serf is a 5-kernel, it follows that every m -partite tournament has a 5-kernel. Besides proving the existence of 4-kernels in m -partite tournaments, we give a sufficient condition for an m -partite tournament to have a 3-kernel. We choose to include the proof for $k \geq 4$ since it is very similar to the proof of the case where $k = 4$, and we also think that it is enlightening in the use of the k -closure.

Theorem 4.2. *Let $T = (V_1, V_2, \dots, V_m)$ be an m -partite tournament, then T has a k -kernel for every $m \geq 2$, $k \geq 4$. If every directed cycle of length 4 in T intersects 4 different classes of T , then T has a 3-kernel for every $m \geq 2$.*

Proof. Let T be an m -partite tournament with $m \geq 2$. Let $k \geq 3$ be an integer and consider the $(k-1)$ -closure of T . In virtue of Lemma 3.2, to prove the result it suffices to show that $\mathcal{C}_{k-1}(T)$ has a kernel, for that aim we will prove that every directed cycle in $\mathcal{C}_{k-1}(T)$ has at least one symmetrical arc, this will be done by induction on the length of the cycle. For the base case let us consider a directed cycle \mathcal{C} of length 3 in $\mathcal{C}_{k-1}(T)$. We have three possible cases, that the three vertices of \mathcal{C} are in distinct classes of T , that two vertices are in the same class and the third one is in a different class, and that the three vertices are in the same class of T .

Case 1. If the three vertices of the cycle are in distinct classes of T , let us say, $\mathcal{C} = (x, y, z, x)$ with $x \in X$, $y \in Y$, $z \in Z$ and $X \neq Y$, $X \neq Z$, $Y \neq Z$.

Case 1.1. If any of the arcs in \mathcal{C} is not an arc of T , for instance $(x, y) \notin A(T)$, since T is a multipartite tournament and x, y are in distinct classes of T , it must exist the arc $(y, x) \in A(T)$ and therefore also $(y, x) \in A(\mathcal{C}_{k-1}(T))$, and thus this is the symmetrical arc we want.

Case 1.2. If the three arcs of \mathcal{C} are arcs of T , in particular we have that $(x, y), (y, z) \in A(T)$, such arcs conform an xz -directed walk of length $2 \leq k-1$ and thence $(x, z) \in A(\mathcal{C}_{k-1}(T))$. We can conclude that the arc $(z, x) \in A(\mathcal{C})$ is symmetrical.

Case 2. If $\mathcal{C} = (x_1, y, x_2, x_1)$ with $x_1, x_2 \in X$, $y \in Y$, $Y \neq X$.

Case 2.1. If $(x_1, y) \notin A(T)$, then $(y, x_1) \in A(T)$, because T is a multipartite tournament and x_1 and y are in distinct classes of T . Thus, the arc $(x_1, y) \in A(\mathcal{C})$ is symmetrical in $\mathcal{C}_{k-1}(T)$. A similar argument can be applied for $(y, x_2) \notin A(D)$.

Case 2.2. If $(x_1, y), (y, x_2) \in A(T)$, then $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$, and thus the arc $(x_2, x_1) \in A(\mathcal{C})$ is a symmetrical arc.

Case 3. If $\mathcal{C} = (x_1, x_2, x_3, x_1)$ with $x_i \in X$, $i \in \{1, 2, 3\}$, following from the fact that T is an m -partite tournament and x_1, x_2 are in the same class of T , we have that $(x_1, x_2) \notin A(T)$, but as $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$, then there exists a x_1x_2 -directed path of length less than or equal to $k-1$ in T , namely $\mathcal{D} = (u_0 = x_1, u_1, \dots, u_{p-1}, u_p = x_2)$, where $p \leq k-1$ and $u_1, u_{p-1} \notin X$ (the length of \mathcal{D} can be 2, in that case $u_1 = u_{p-1}$). Since $u_1 \notin X$, then $(u_1, x_3) \in A(T)$ or $(x_3, u_1) \in A(T)$.

Case 3.1. If $(u_1, x_3) \in A(T)$, then $d_T(x_1, x_3) = 2$ and hence $(x_1, x_3) \in A(\mathcal{C}_{k-1}(T))$,

which turns out to be the wanted symmetrical arc.

Case 3.1. If $(x_3, u_1) \in A(T)$, then $(x_3, u_1) \cup (u_1 \mathcal{D} x_2)$ is an $x_3 x_2$ -directed path of length $p \leq k - 1$ and in this way $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ resulting in the symmetrical arc we have been looking for.

Let us assume for the inductive step that every directed cycle of $\mathcal{C}_{k-1}(D)$ of length less than or equal to n has at least one symmetrical arc, and let \mathcal{C} be a directed cycle of length $n \geq 4$ in $\mathcal{C}_{k-1}(T)$. Let us observe three (arbitrarily chosen) consecutive vertices of \mathcal{C} , we have 5 cases and several subcases.

Case 1. The considered segment of the directed cycle is (x, y, z) , with $x \in X$, $y \in Y$, $z \in Z$ and $X \neq Y$, $X \neq Z$, $Y \neq Z$.

Case 1.1 If any of the arcs in \mathcal{C} is not an arc of T , for instance $(x, y) \notin A(T)$, since T is a multipartite tournament and x, y are in distinct classes of T , it must exist the arc $(y, x) \in A(T)$ and therefore also $(y, x) \in A(\mathcal{C}_{k-1}(T))$, and thus this is the symmetrical arc we want.

Case 1.2. If $(x, y), (y, z) \in A(T)$ thence $(x, z) \in A(\mathcal{C}_{k-1}(T))$.

Case 1.2.1. If $(x, z) \notin A(T)$, then $(z, x) \in A(T)$ and together with the arc (x, y) we can deduce the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$, which would be a symmetrical arc of \mathcal{C} .

Case 1.2.2. Thus, we can suppose that $(x, z) \in A(T)$. We have that $(x, z) \cup (z \mathcal{C} x)$ is a directed cycle of length $n - 1$ which has a symmetrical arc for the induction hypothesis. If the symmetrical arc that exists for the induction hypothesis is in $z \mathcal{C} x$, then it is a symmetrical arc of \mathcal{C} . Let us assume then that the symmetrical arc is (x, z) . Since $(z, x) \notin A(T)$, but $(z, x) \in A(\mathcal{C}_{k-1}(T))$, a zx -directed path of length greater than or equal to 2 but less than or equal to $k - 1$ exists in T , namely $\mathcal{D} = (z = u_0, u_1, \dots, u_{p-1}, u_p = x)$ with $p \leq k - 1$.

Case 1.2.2.1. If $u_1 \notin Y$, then $(u_1, y) \in A(T)$ or $(y, u_1) \in A(T)$. In the former case, together with the arc (z, u_1) it follows the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$, that would be a symmetrical arc in \mathcal{C} . When $(y, u_1) \in A(T)$, $(y, u_1) \cup (u_1 \mathcal{D} x)$ is a yx -directed path of length $p \leq k - 1$, which implies that $(y, x) \in A(\mathcal{C}_{k-1}(T))$ and this would be the symmetrical arc of \mathcal{C} we have been looking for.

Case 1.2.2.2. If $u_1 \in Y$ and $\mathbf{p} > 2$, then $k \geq 4$ and necessarily $u_2 \neq x$ and $u_2 \notin Y$, so $(u_2, y) \in A(T)$ or $(y, u_2) \in A(T)$. For the former case we can consider the directed path (z, u_1, u_2, y) in T and, since $k \geq 4$, the existence of this path implies the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$; for the latter case the directed path $(y, u_2) \cup (u_2 \mathcal{D} x)$ is in T and has length $p - 1 < k - 1$ so the existence of the arc $(y, x) \in A(\mathcal{C}_{k-1}(T))$ can be deduced.

Finally, if $u_1 \in Y$ and $\mathbf{p} = 2$, then $\mathcal{D} = (z, u_1, x)$ and it follows that $k \geq 4$, because for $k = 3$ we are assuming that every directed cycle of length 4 in T intersects 4 distinct classes of T but (x, y, z, u_1, x) is a directed cycle of length 4 in T that intersects only 3 different classes of T . So, $(x, y) \cup \mathcal{D}$ is a zy -directed path of length $3 \leq k - 1$ and hence

$(z, y) \in A(\mathcal{C}_{k-1}(T))$ is the desired symmetrical arc.

Case 2. The considered segment of \mathcal{C} is (x_1, x_2, x_3) with $x_1, x_2, x_3 \in X$. Since $(x_1, x_2) \notin A(T)$, but $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$, there must exist an x_1x_2 -directed path of length $2 \leq p \leq k-1$ in T , namely $\mathcal{D} = (x_1 = u_0, u_1, \dots, u_{p-1}, u_p = x_2)$. Let us observe that $(u_0, u_1) \in A(T)$, so $u_1 \notin X$ and hence $(x_3, u_1) \in A(T)$ or $(u_1, x_3) \in A(T)$.

Case 2.1. If $(x_3, u_1) \in A(T)$, then $(x_3, u_1) \cup (u_1\mathcal{D}x_2)$ is an x_3x_2 -directed path of length $p \leq k-1$ in T , and the existence of the arc $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ now follows, moreover, this arc is a symmetrical arc of \mathcal{C} .

Case 2.2. If $(u_1, x_3) \in A(T)$, together with the arc (x_1, u_1) we can deduce that $(x_1, x_3) \in A(\mathcal{C}_{k-1}(T))$ and, analogously to Case 1, $(x_1, x_3) \cup (x_3\mathcal{C}x_1)$ is a directed cycle of length $n-1$ in $\mathcal{C}_{k-1}(T)$ which has at least one symmetrical arc by the induction hypothesis; if the symmetrical arc is other than (x_1, x_3) then \mathcal{C} would have a symmetrical arc, so we can suppose that $(x_3, x_1) \in A(\mathcal{C}_{k-1}(T))$ from where we can deduce the existence of $\mathcal{E} = (x_3 = v_0, v_1, \dots, v_{q-1}, v_q = x_1)$, an x_3x_1 -directed path of length $2 \leq q \leq k-1$ in T . Since $(x_3, v_1) \in A(T)$, it follows that $v_1 \notin X$ and thus $(v_1, x_2) \in A(T)$ or $(x_2, v_1) \in A(T)$.

Case 2.2.1. If $(v_1, x_2) \in A(T)$ then (x_3, v_1, x_2) is a directed path in T and therefore $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ is a symmetrical arc in \mathcal{C} .

Case 2.2.2. If $(x_2, v_1) \in A(T)$ we can consider the x_2x_1 -directed path $(x_2, v_1) \cup (v_1\mathcal{E}x_1)$ of length $q \leq k-1$, which implies the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$, a symmetrical arc of \mathcal{C} .

Case 3. The considered segment of the directed cycle is (x_1, x_2, y) with $x_1, x_2 \in X$, $y \in Y$, $X \neq Y$. If $(y, x_2) \in A(T)$ then the cycle has a symmetrical arc. Let us assume that $(x_1, y), (y, x_2) \in A(T)$, and since x_1 and y are in distinct classes of T , then one of the two possible arcs between them must exist in T .

Case 3.1. If $(y, x_1) \in A(T)$, then together with the arc (x_2, y) the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be derived, and this is the desired symmetrical arc.

Case 3.2. Let us suppose then that $(x_1, y) \in A(T)$, analogously to Case 1, $(x_1, y) \cup (y\mathcal{C}x_1)$ is a directed cycle of length $n-1$ which has a symmetrical arc from the induction hypothesis, moreover, such arc must be (x_1, y) or we would have the symmetrical arc in \mathcal{C} we have been looking for. Since $(y, x_1) \in \mathcal{C}_{k-1}(T)$ but $(y, x_1) \notin A(T)$, then a yx_1 -directed path of length $2 \leq p \leq k-1$ exists in T , namely $\mathcal{D} = (y = u_0, u_1, \dots, u_{p-1}, u_p = x_1)$. But, as $(u_{p-1}, x_1) \in A(T)$, then $u_{p-1} \notin X$ and therefore $(u_{p-1}, x_2) \in A(T)$ or $(x_2, u_{p-1}) \in A(T)$.

Case 3.2.1 In the former case, $(y\mathcal{D}u_{p-1}) \cup (u_{p-1}, x_2)$ is a yx_2 -directed path of length $p \leq k-1$ which implies the existence of $(y, x_2) \in A(\mathcal{C}_{k-1}(T))$, resulting in a symmetrical arc of \mathcal{C} .

Case 3.2.2 In the latter case, together with the arc (u_{p-1}, x_1) the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be inferred, and this is the wanted symmetrical arc.

Case 4. The considered segment of \mathcal{C} is (y, x_1, x_2) with $x_1, x_2 \in X$, $y \in Y$, $X \neq Y$. Since the length of \mathcal{C} is greater than or equal to 4, then the cycle has at least another vertex,

namely z , such that (x_1, x_2, z) is also a segment of \mathcal{C} . If $z \in Z \neq X$ then we have the same situation as in Case 3. If $z \in X$ then we have the same situation as in Case 2. In any case, we know that \mathcal{C} has at least one symmetrical arc.

Case 5. The considered segment in the directed cycle is (x_1, y, x_2) with $x_1, x_2 \in X, y \in Y, X \neq Y$. If $(y, x_1) \in A(T)$ or $(x_2, y) \in A(T)$ then the cycle has a symmetrical arc. Let us assume that $(x_1, y), (y, x_2) \in A(T)$, from where we can infer that $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$ and $(x_1, x_2) \cup (x_2 \mathcal{C} x_1)$ is a directed cycle of length $n - 1$ in $\mathcal{C}_{k-1}(T)$ which has at least one symmetrical arc from the induction hypothesis. Such arc must be (x_1, x_2) or the existence of a symmetrical arc in \mathcal{C} would be already proven. Therefore $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ and an $x_2 x_1$ -directed path $\mathcal{D} = (x_2 = u_0, u_1, \dots, u_{p-1}, u_p = x_1)$ must exist in T with $2 \leq p \leq k - 1$.

Case 5.1. If $p = 2$ then $k \geq 4$ because (x_1, y, x_2, u_1, x_1) is a directed cycle of length 4 intersecting only 3 distinct classes of T , which cannot occur for $k = 3$, so (x_2, u_1, x_1, y) is an $x_2 y$ -directed path of length $3 \leq k - 1$ and then $(x_2, y) \in A(\mathcal{C}_{k-1}(T))$ is a symmetrical arc of \mathcal{C} .

Case 5.2. If $p > 2$ then there is at least one index $1 \leq i \leq p - 1$ such that $u_i \notin Y$ and hence $(u_i, y) \in A(T)$ or $(y, u_i) \in A(T)$.

Case 5.2.1. In the former case, $(x_2 \mathcal{D} u_i) \cup (u_i, y)$ is an $x_2 y$ -directed path of length less than or equal to $p \leq k - 1$ and therefore $(x_2, y) \in A(\mathcal{C}_{k-1}(T))$ is the desired symmetrical arc in \mathcal{C} .

Case 5.2.2. For the case when $(y, u_i) \in A(T)$ we can consider the following $y x_1$ -directed path of length less than or equal to $p \leq k - 1$: $(y, u_i) \cup (u_i \mathcal{D} x_1)$ from which the existence of the arc $(y, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be deduced and is, in fact, a symmetrical arc of \mathcal{C} .

Since the cases are exhaustive, the desired result follows from the principle of mathematical induction and an application of Theorem 3.4. □

Corollary 4.3. *If T is an m -partite tournament with $m \geq 2$ and no directed cycles of length 4, then T has a 3-kernel.*

Corollary 4.4. *If T is an m -partite tournament with $m \geq 2$, then T has a k -solution for every $k \geq 4$. If every directed cycle of length 4 intersects four different classes of T , then T has a 3-solution for every $m \geq 2$.*

Proof. It follows from Remark 2.4 and Theorem 4.2. □

Another family of multipartite digraphs, generalizing multipartite tournaments, has been widely studied.

Definition 4.5. *A semicomplete multipartite digraph is a biorientation of a complete multipartite graph. If D is a semicomplete multipartite digraph, it is usually denoted by $D = (V_1, V_2, \dots, V_m)$, where $\{V_1, V_2, \dots, V_m\}$ is the vertex partition of D ; the sets $V_i, i \in \{1, \dots, m\}$ are called the classes of D . If D has m classes then it is called a semicomplete m -partite digraph.*

We can easily observe that exactly the same proof of Theorem 4.2 can be used to prove the following result.

Corollary 4.6. *Let $D = (V_1, V_2, \dots, V_m)$ be a semicomplete m -partite digraph, then D has a k -kernel for every $m \geq 2, k \geq 4$. If every directed cycle of length 4 in D intersects 4 different classes of D , then D has a 3-kernel for every $m \geq 2$.*

Thus, also Corollaries 4.3 and 4.4 can be stated for semicomplete multipartite digraphs.

Before making a full study of the case $k = 3$, let us observe that for case $k = 2$ we have the usual notion of kernel in Berge’s sense and we know that every bipartite digraph has a kernel, so every bipartite tournament has a kernel. For m -partite tournaments with $m \geq 3$ we can always find an m -partite tournament without a kernel, namely, a tournament without vertices with exdegree equal to 0 on m vertices. As a matter of fact, since absorbency is 1-absorbency, in this case an m -partite tournament T will have a kernel if, and only if, there is a class X of T such that for every vertex $v \in V(T) \setminus X$ there exists a vX -arc in T . In this case X will be the desired kernel. If we regard a tournament of order m as an m -partite tournament this will happen if, and only if, there is a vertex of exdegree equal to 0 in the tournament.

For the case $k = 3$ we can also construct an m -partite tournament for every $m \geq 2$ without a 3-kernel. For $m = 2$ it suffices to consider the directed cycle of length 4, C_4 , or the strong orientation of $K_{3,3}$ presented next.

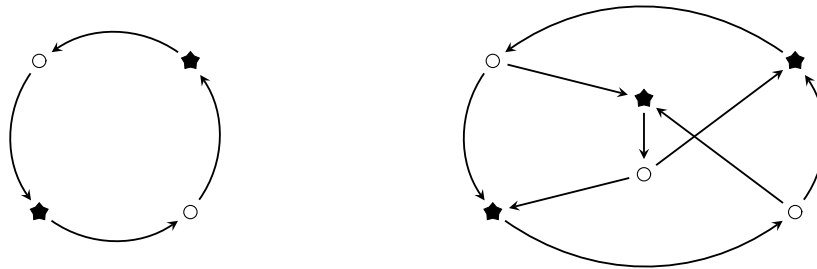


Figure 1: Examples of bipartite tournaments without a 3-kernel

In both digraphs depicted in Figure 1 the partition is given by the circular white vertices and the black starred ones. In both cases, every vertex 2-absorbs every other vertex except for its out-neighborhood, so, the vertices in its out-neighborhood are not 2-absorbed nor can be added to the 3-kernel.

For $m \geq 3$, we can define the m -partite tournament $T_{C_4,m}$ with vertex set $V(T_{C_4,m}) = \{1, 1', 2, 2'\} \cup \{3, \dots, m\}$ and arc set $A(T_{C_4,m}) = \{(1, 2), (2, 1'), (1', 2'), (2', 1)\} \cup \bigcup_{i=3}^m \{(i, 1), (i, 1'), (i, 2), (i, 2')\} \cup \bigcup_{3 \leq i < j} \{(j, i)\}$ with vertex partition given by $V(T_{C_4,m}) = \bigcup_{i=1}^2 \{i, i'\} \cup \bigcup_{i=3}^m \{i\}$.

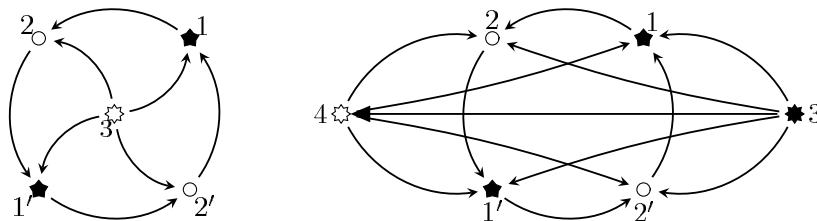


Figure 2: $T_{C_{4,3}}$ and $T_{C_{4,4}}$

As an obvious consequence of Theorem 4.2, every example of an m -partite tournament T without a 3-kernel must have a copy of C_4 as a subdigraph, and this copy of C_4 must not intersect 4 distinct classes of T . Unfortunately, the sufficient condition presented in Theorem 4.2 for an m -partite tournament to have a 3-kernel is not necessary. The next figure shows an example of a bipartite tournament with a copy of C_4 which clearly does not intersect 4 different classes of T and with a 3-kernel.

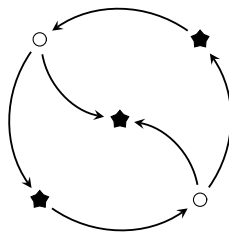


Figure 3: An example of a bipartite tournament with a copy of C_4 and a 3-kernel.

The central vertex in the bipartite tournament of Figure 3 is a 3-kernel for the tournament.

But, let us observe the structure of a 3-kernel in an m -partite tournament T . Obviously, since vertices in distinct classes are always adjacent, not only for $k = 3$ but for every k , a k -kernel must be contained in a single class of T . The next proposition explore a necessary condition for an m -partite tournament to have a 3-kernel.

Proposition 4.7. *If T is an m -partite tournament with a 3-kernel N , then for every $v \in N$, the set $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$ where X is the class of T which contains N .*

Proof. Let T be an m -partite tournament with 3-kernel N and let X be the class of T that contains N . Let $v \in N$ be an arbitrary vertex. Clearly the in-neighborhood of v is 2-absorbed by v , so let us think of a vertex $u \in V(T) \setminus (X \cup N^-(v))$, then since T is a multipartite tournament, $u \in N^+(v)$, and since N is a 3-kernel of T , there must exist a vertex $w \in N$ such that w 2-absorbs u . If $w = v$ we are done.

For the bipartite case, every vertex in $V(T) \setminus X$ can only be 2-absorbed at distance 1 by N , so if $w \neq v$, then the existence of the directed path (v, u, w) would contradict the 3-independence of N . So for the bipartite case it follows that $u \in N^-(v)$.

For $m \geq 3$, let us assume that $v \neq w$; if $u \in N^-(w)$, then (v, u, w) would be an NN -directed path of length 2 in T and N would not be 3-independent, so $u \in N^+(w)$ and then there exists a vertex $z \notin X$ such that (u, z, w) is a directed path in T . But, as $z \notin X$, then $(v, z) \in A(T)$ or $(z, v) \in A(T)$; the former case cannot occur because the directed path (v, z, w) would contradict the 3-independence of N . So, $(z, v) \in A(T)$ and (u, z, v) is a directed path of length 2 in T , so $\{v\}$ 2-absorbs u . Since u was chosen arbitrarily in $V(T) \setminus (X \cup N^-(v))$, then $\{v\}$ is an absorbent set for $T - (X \setminus \{v\})$. \square

Well, at this point the obvious question arise, is the converse of Proposition 4.7 also true? That is, if T is an m -partite tournament with a vertex v such that $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$, where X is the class of T that contains v , then T has a 3-kernel? In that case we would have a characterization of multipartite tournaments with a 3-kernel. Not only the answer to this question is yes, we also have a third equivalence inspired in the observations made about directed cycles of length 4 when we were looking for a 3-kernel.

Theorem 4.8. *Let T be an m -partite tournament with $m \geq 2$, then the following assertions are equivalent.*

1. T has a 3-kernel.
2. There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v , $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$.
3. There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v , $\{v\}$ 2-absorbs in T every $x \in \{v\} \cup (T \setminus X)$ such that x is in a directed cycle of length 4 of T .

Proof. For (i) implies (ii) we just have to choose an arbitrary vertex in the 3-kernel, the result follows from Proposition 4.7.

Trivially (ii) implies (iii).

For (iii) implies (i), let v be a vertex that fulfills the property stated in (iii). Let $R = \{v\} \cup \{u \in V(T) \mid d_T(u, v) \leq 2\}$, that is, the set of vertices that are 2-absorbed by v . As a consequence of the choice of v and Corollary 4.3, if $S := T \setminus R$ is non empty, it has a 3-kernel, say N (if it is empty, then $\{v\}$ is a 3-kernel of T). If $N \cup \{v\}$ is a 3-independent set in T , then since N is 2-absorbent in S and $\{v\}$ 2-absorbs every vertex in R , then $N \cup \{v\}$ is a 2-absorbing set and thus the desired 3-kernel of T . If $N \cup \{v\}$ is not 3-independent then there are two possibilities, that $N \subseteq X$ or that $N \subseteq Y$ for a class $Y \neq X$ of T . In either case, in virtue of Claim 4.9, we can assume without loss of generality that N is 3-independent in T . Our next claim will be proved after completing the proof of the theorem.

Claim 4.9. *If the 3-kernel N that exists for S is not 3-independent in T , we can choose $K \subseteq N$ such that K is 3-independent in T , every vertex 2-absorbed by N in S is also absorbed by K and every vertex in $N \setminus K$ is 2-absorbed by K .*

If $N \subseteq X$, since N is 3-independent, $N \cup \{v\}$ is not 3-independent but $v \in X$, there must be a vx -directed path of length two or an xv -directed path of length two in T for some $x \in N$. But the latter case cannot occur, or x would be 2-absorbed by v and it would belong to R . So, the former case occurs and v is 2-absorbed by N in T . If we prove that $N^-(v) \subseteq N^-(x)$, then every vertex 2-absorbed by v will be also 2-absorbed by x and then N will be the desired 3-kernel. Let $y \in N^-(v)$ be chosen arbitrarily. Since T is a multipartite tournament, $x \in X$ and $y \notin X$, then $(x, y) \in A(T)$ or $(y, x) \in A(T)$; but if $(x, y) \in A(T)$, then (x, y, v) would be an xv -directed path of length 2, contradicting that $x \notin R$, so $(y, x) \in A(T)$ and then $y \in N^-(x)$, so $N^-(v) \subseteq N^-(x)$.

If $N \subseteq Y$ for a class $Y \neq X$ of T , then for every $x \in N$, $(v, x) \in A(T)$, so v and $N^-(v)$ are 2-absorbed by N in T . Let u be a vertex in R such that $(v, u) \in A(T)$, then there is a vertex $w \in N^-(v)$ such that $(u, w) \in A(T)$. If $w \notin Y$ and $x \in N$, then $(w, x) \in A(T)$ or $(x, w) \in A(T)$. If $(x, w) \in A(T)$, since $w \in N^-(v)$, x would be 2-absorbed by v and thus it would be an element of R , which is not the case. It follows that $(w, x) \in A(T)$, then (u, w, x) is a directed path in T and u is 2-absorbed by N in T . If $w \in Y$, then $u \notin Y$, and then, if $x \in N$, $(u, x) \in A(T)$ or $(x, u) \in A(T)$. If $(x, u) \in A(T)$, then (v, x, u, w, v) is a directed cycle of length 4 in T , and by the choice of v fulfilling the conditions in (iii), x would be 2-absorbed by v and $x \in R$, which would be a contradiction. Thus, $(u, x) \in A(T)$ and then N 2-absorbs every vertex in R and is the desired 3-kernel. \square

Proof of Claim 4.9

Let us recall that $S := T - R$. In virtue of Proposition 4.7, if $N \subseteq Z$ for a class Z of T , then every vertex in N 2-absorbs every vertex in $S \setminus Z$, so we just have to find a subset $K \subseteq N$ that is 3-independent and 2-absorbs every vertex in $Z \setminus (R \cup N)$ and every vertex in $N \setminus K$.

Let $x \in Z \setminus (R \cup N)$ be an arbitrarily chosen vertex, since N is a 3-kernel for S and $x \in Z \setminus N$, then $d_S(x, u) = 2$ for some vertex $u \in N$, so there is a vertex $y \in V(T) \setminus (Z \cup R)$ such that (x, y, u) is a directed path in T . Since $y \notin Z$, then for every other vertex $w \in N$, $(w, y) \in A(T)$ or $(y, w) \in A(T)$. But, if $(w, y) \in A(T)$, then $d_S(w, u) = 2$, contradicting the 3-independence of N in S . Hence, $(y, w) \in A(T)$ and then every vertex in $Z \setminus (R \cup N)$ is 2-absorbed by every vertex in N .

Let us prove by means of mathematical induction on the cardinality of N that we can always choose a subset $K \subseteq N$ such that K is 3-independent and 2-absorbs in T every vertex in $N \setminus K$. We will consider two cases for the base case. If $|N| = 1$, then $K = N$ will work.

If $|N| = 2$ and N is 3-independent in T we are done; otherwise $N = \{u, w\}$ and it follows from the 3-dependence of N that $d_S(u, w) = 2$ or $d_S(w, u) = 2$, let us assume without loss

of generality that $d_S(u, w) = 2$, then u is 2-absorbed by $\{w\}$ and hence $\{w\} = K$.

Suppose as induction hypothesis that if $|N| < n$ we can find $K \subseteq N$ with the desired property, and let N be a 3-kernel for S with cardinality n . If N is 3-independent we are done. Otherwise, there exists vertices $u, w \in N$ such that $d_S(u, w) = 2$, so u is 2-absorbed by N and $N' := N \setminus \{u\}$ is a 3-kernel of $S - u$, so from the induction hypothesis there exists $K' \subseteq N'$ such that K' is 3-independent in T and absorbs every vertex in $N' \setminus K'$.

We have three possible cases.

Case 1. If K' 2-absorbs u , then $K = K'$ is the desired set.

Case 2. If K' does not absorb u and $K' \cup \{u\}$ is 3-independent in T , then $K = K' \cup \{u\}$ is the subset we have been looking for.

Case 3. If K' does not 2-absorb u and $K' \cup \{u\}$ is not 3-independent, since N' 2-absorbs u , then $K' \subset N'$ and thus $|K' \cup \{u\}| \leq |N'| < |N|$. We also have that $K' \cup \{u\}$ is a 3-kernel of $S - (N \setminus (K' \cup \{u\}))$, so it follows from the induction hypothesis that there exists $K \subset K' \cup \{u\}$ such that K is 3-independent and 2-absorbs in T every vertex in $(K' \cup \{u\}) \setminus K$.

Finally, we need K to 2-absorb in T every vertex in $N \setminus (K' \cup \{u\})$ but, if $x \in N \setminus (K' \cup \{u\})$, for the choice of K' there exist $z \in K'$ and $y \in V(T) \setminus Z$ such that (x, y, z) is a directed path in T . If k is an arbitrary vertex in K , then $(k, y) \in A(T)$ or $(y, k) \in A(T)$. If $(k, y) \in A(T)$, then $d_T(k, z) = 2$, contradicting the 3-independence of K' in T . So, it must be the case that $(y, k) \in A(T)$, and thence K 2-absorbs x in T .

Since the cases are exhaustive, the result follows from the principle of mathematical induction. \square

Again, we may observe that the proofs of Theorem 4.8 and Claim 4.9 work for analogous statements about semicomplete multipartite digraphs.

Corollary 4.10. *Let T be an m -partite tournament with $m \geq 2$, then the following assertions are equivalent.*

1. T has a 3-solution.
2. There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v , $\{v\}$ is a 2-dominating set of $T - (X \setminus \{v\})$.
3. There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v , $\{v\}$ 2-dominates in T every $x \in \{v\} \cup (T \setminus X)$ such that x is in a directed cycle of length 4 of T .

Proof. It follows from Remark 2.4 and Theorem 4.8. \square

5. Conclusion

We successfully used the k -closure of a digraph D to prove that every semicomplete multipartite digraph has a k -kernel for every integer $k \geq 4$, and to characterize the semicomplete multipartite digraphs with a 3-kernel. But the k -closure may be used not only in the context of k -kernels, it is clear from the proof of Lemma 3.2 that it may be useful in any problem involving k -independence, l -absorbency or l -domination, turning such problems into problems of independence, absorbency or domination respectively.

Following the steps proposed by the study of k -kings, it would be an interesting problem to give sufficient conditions for a multipartite tournament (or semicomplete multipartite digraph) to have at least n k -kernels or to characterize the multipartite tournaments (semicomplete multipartite digraphs) having exactly n k -kernels. Of course, as a generalization of k -kings, this may be a very difficult problem, but perhaps the k -closure may be useful for this task.

After quite a few time of intermitent activity on the subject of k -kernels, in the past few years some large families of digraphs have been successfully studied, and general and promising results have been obtained. We hope that characterization theorems such as Theorem 4.8 or tools like the k -closure may be an impulse to the investigation in this beautiful field.

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