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Level Hypergraphs-II

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Abstract

In a previous paper [14] we introduced a way of relating any hypergraph with a simpler one, which retains its edge-structure but has often much less vertices. This allows results in several branches of hypergraph theory.

In this paper we define the tool again, and give several examples of its use. While some of the results obtained apply to different classes of hypergraphs than the original statements, others are generalisations of classical theorems.

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1. Introduction

Given a hypergraph $H = (E_1, \ldots, E_m)$ on a set V, its level-hypergraph is the result of identifying all vertices which belong to exactly the same edges. This new hypergraph has the same edge-structure as the original one, but may have less vertices. Since several hypergraph invariants are preserved, it is possible to obtain new results from theorems given in terms of order or rank. These new statements are given in terms of edge-structure; some are generalisations, and other are independent from the original results.

The concept was defined by B. D. Acharya in [1], and independently (30 years later) by us in [14], where we also proved some properties regarding level hypergraphs and gave examples of its utility to "emulate" known theorems. When needed, results appearing in [14] will be used without proof.

In this paper we describe the method and give examples of its application to classical results, most of them appearing in [4].

For general concepts on graphs and hypergraphs we refer the reader to [3] and [4] respectively.

2. Foundations

Definition 2.1. *Given a hypergraph $H = (E_1, \ldots, E_m)$, its natural partition $P = \{P_1, \ldots, P_l\}$ is the partition of the vertex set $V(H)$ of H defined by the equivalence relation $\approx$, specified by the rule $x \approx y \Leftrightarrow H(x) = H(y)$, where, for any vertex w, $H(w)$ denotes the set of edges containing w. The elements of P are called levels of H, and for any edge E of H, the sets $\{P_i \subset E \mid P_i \in P\}$, called levels of E, form the natural partition of E.*

Definition 2.2. *Given a hypergraph $H = (E_1, \ldots, E_m)$ with natural partition $P = \{P_1, \ldots, P_l\}$, the level hypergraph $L_H = (E'_1, \ldots, E'_m)$ of H is the hypergraph resulting from deleting every vertex but one from each level P_i of P. In other words, we consider a set $S = \{x_1, \ldots, x_l \mid x_i \in P_i \ \forall i \in \{1, \ldots, l\}\}$ and take $L_H = H[S]$, the subhypergraph induced by S. It is clear that L_H is well defined, that is, it does not matter which vertex from each level is kept, for all of them play an equivalent role.*

The notions above were independently given by Acharya in [1] (and that of natural partition by us in [15]), where he considers hypergraphs with isolated vertices (that is, vertices which belong to no edge). Here we follow [4], so "isolated" vertices have a loop. The only difference is that in [1] all isolates belong to one level, while in this paper each "isolate" belongs to its own level. All results in this paper apply directly or are easily adapted to hypergraphs with (true) isolates.

Since every edge E of H has at least one vertex, it contains at least one level, so it induces an edge E' in L_H; then both H and L_H have the same number of edges. In the same way, E_i and E'_i contain the same number of levels and $E_i \cap E_j \neq \emptyset$ if, and only if, $E'_i \cap E'_j \neq \emptyset$. This implies that H is simple if, and only if, L_H is simple, that H has repeated edges if, and only if, L_H does too, and that $\nu(H) = \nu(L_H)$, where $\nu(H)$ is the maximum cardinality of a matching in H. Notice also that every hypergraph H such that every level of its natural partition has but one vertex, is a level hypergraph (of itself, to begin with), so we can speak of a "level hypergraph" without considering any "original" hypergraph.

Any induced subhypergraph of a level hypergraph is itself a level hypergraph; in fact, a hypergraph is a level hypergraph if, and only if, all of its induced subhypergraphs are level hypergraphs. Given a hypergraph H, the level hypergraph of any induced subhypergraph of H is an induced subhypergraph of L_H. However, a partial hypergraph of a level hypergraph is not always a level hypergraph (see Observation 7.11).

Definition 2.3. *Given a hypergraph H and a vertex $x \in V(H)$, the edge degree $d_H(x)$ of x in H is the number of edges in H containing x. From now on, it will be called simply degree.*

Observe that all vertices belonging to a given level P_i of H have the same degree, as well as the vertex corresponding to that level in L_H. In particular, maximum and minimum degrees are preserved: $\Delta(H) = \Delta(L_H)$ and $\delta(H) = \delta(L_H)$.

Given an edge E of a hypergraph H, the corresponding edge of L_H will be called E', and an edge of L_H will always be written with an apostrophe. We will use the same symbol for a level of H and its corresponding level in L_H. We consider $V(L_H) = \{x_1, ..., x_l \mid x_i \in P_i \ \forall i \in \{1, ..., l\}\}$ and call x_i the *representative* of P_i.

Unless stated otherwise, we will use the following standard notation for any hypergraph H:

P = natural partition of H; $V(H)$ = set of vertices of H; $n = |V(H)|$ = order of H; $m = |H|$; $l = |P|$; r_E = number of levels contained in edge E; $r = \max_{E\in H} |E|$ = rank of H; $s = \min_{E\in H} |E|$ = anti-rank of H; $r' = \max_{E\in H} r_E$; $s' = \min_{E\in H} r_E$; H^* = dual of H; $H[A]$ = subhypergraph induced by A; $\widetilde{H}$ = partial hypergraph of H; $H(x) = \{E \in H \mid x \in E\}$; $\tau = \min\{|T|$ such that T is a transversal set of $H\}$; $\tau' = \max\{|T|$ such that T is a transversal set of H not containing properly any other transversal$\}$; $\alpha = \max\{|T|$ such that T is a strongly stable set of $H\}$; $\nu = \max\{|T|$ such that T is a matching of $H\}$; $\rho = \min\{|T|$ such that T is a covering of $H\}$; $N_H(x)$ = the set of vertices adjacent to x in $H = V(H(x)) \setminus \{x\}$; $N_H(A)$ = the set of vertices adjacent to any vertex $x \in A$; $[a]$ = largest integer not larger than a; $[a]^*$ = smallest integer not smaller than a. We make no explicit difference between levels of H and levels of L_H, and for any edge E of H, the edge of L_H induced by E is called E'.

In [1] (quoted in [14]) appears an easy way of characterising a level hypergraph: A given hypergraph H is a level hypergraph if, and only if, for every set $\{x, y\} \subset V(H)$, there is an edge E in H such that $x \in E$ and $y \notin E$, or $y \in E$ and $x \notin E$. Equivalently, H is a level hypergraph if, and only if, for every $x \in V(H)$, the natural map $x \mapsto H(x)$ is a bijection. Level hypergraphs may be also characterised by their incidence matrix: A given hypergraph H is a level hypergraph if, and only if, its incidence matrix has no repeated rows. ($\Leftrightarrow$ the incidence matrix of H^* has no repeated columns).

Some examples of level hypergraphs are the following: r-complete hypergraphs K_n^r, hereditary hypergraphs, projective planes, and k-partite complete hypergraphs $K^r_{n_1,...,n_k}$ if no more than one element of the partition has only one vertex. This can be easily verified by means of the characterisations given above. Nevertheless, the last statement will be proved in Proposition 2.11.

Since many hypergraph invariants are equal in both H and L_H, it is possible to consider a theorem regarding some of them whose conditions involve cardinality, then "translate" it to a result stated in terms of edge structure. This method enables generalisations of classical theorems, although often the new results apply to a different class of hypergraphs than the original ones.

Observation 2.4. *Let a, b, c, d be positive integers such that $a \geq b$, $c \geq d$, $c \geq a$ and $d \geq b$. If $c - a \geq d - b$, then $\binom{c}{d} \geq \binom{a}{b}$. This elementary observation shows that many emulated theorems give better bounds than the original ones (although they do not always apply to the same class of hypergraphs). Moreover, notice that for every edge E of a given hypergraph H we have that $l - r_E \leq n - |E|$.*

Theorem 2.5. *Every simple hypergraph H with l levels and m edges satisfies:*

1. $\sum_{E\in H} \binom{l}{r_E}^{-1} \leq 1$

2. $m \leq \binom{l}{[l/2]}$.

Proof. If H satisfies the conditions, the order of L_H is l, and for each edge $E' \in L_H$, $|E'| = r_E$. Since H is simple if, and only if, L_H is simple, and both H and L_H have the same number of edges, the result follows from a theorem by Sperner ([24]). □

Theorem 2.5 is an improvement of Sperner's, since for any hypergraph H we have $\binom{l}{r_E} \leq \binom{n}{|E|}$ and $\binom{l}{[l/2]} \leq \binom{n}{[n/2]}$.

Definition 2.6. *A hypergraph H is linear if, and only if, the intersection of any two edges either is empty or has only one vertex. H is level-linear if, and only if, the intersection of any two edges either is empty or has only one level.*

Proposition 2.7. *Let H be a level-linear hypergraph. Then its dual hypergraph H^* is level linear. In fact, H^* is linear but for repeated edges, that is, for every $X, Y \in H^*$, $|X \cap Y| > 1 \Rightarrow V(X) = V(Y)$.*

Proof. Given a hypergraph H, L_H^* is like H^* without repeated edges [14]. Let H be a level-linear hypergraph; then L_H is linear, so [4], Ch. 1, Proposition 3, implies L_H^* is linear. From the first sentence of this paragraph, H^* may have repeated edges, but is otherwise linear. □

Definition 2.8. *A hypergraph H is r'-level-uniform if, and only if, every edge of H has r' levels.*

Theorem 2.9. *Let H be a level-linear hypergraph with l levels and m edges, then $\sum_{E\in H} \binom{r_E}{2} \leq \binom{l}{2}$. If H is also r'-level-uniform, then $m \leq \frac{l(l-1)}{r'(r'-1)}$.*

Proof. If H satisfies the conditions, then L_H is a linear (r'-uniform) hypergraph of order l with m edges. The result then follows from [4], Ch. 1, Theorem 3.

The first statement applies to a wider class of hypergraphs than the original result, since to be level-linear is weaker than to be linear, and the bound given is always better for linear hypergraphs. The second statement gives a better bound than [4], Ch. 1, Theorem 3, but it does not apply to the same class of hypergraphs, for being uniform and being level-uniform are different and non-comparable. □

Definition 2.10. *An r-uniform hypergraph H is r-partite if $V(H) = V_1 \cup \cdots \cup V_r$ is the disjoint union of r sets, and for $i \in \{1, \dots, r\}$ and $E \in H$ we have that $|E \cap V_i| = 1$.*

Proposition 2.11. *An r-partite hypergraph H with $V(H) = V_1 \cup \cdots \cup V_r$ is a level hypergraph if, and only if, no more than one of the sets V_i, $i \in \{1, \dots, r\}$ has only one vertex.*

Proof. If two sets V_i, V_j have only one vertex, then those two vertices belong to every edge, that is, they belong to the same level.

If none of the sets $V_1, \dots, V_r$ has only one vertex, then for every pair of vertices $\{x, y\} \subset V(H)$ there is an edge containing x but not y, and an edge containing y but not x, so H is a level hypergraph. Similarly, if only one set V_i satisfies $|V_i| = 1$, its single element is the only vertex belonging to every edge, so that for every pair of vertices in $V(H)$ there is an edge containing only one of them, which implies that H is a level hypergraph. □

3. Intersecting Families

Definition 3.1. *Let H be a hypergraph. A family of edges $J \subset H$ is intersecting if, and only if, $E \cap F \neq \emptyset$ for every $\{E, F\} \subset J$. The maximum cardinality of an intersecting family in H is denoted $\Delta_0(H)$. When $\Delta_0(H) = m$ we say that H is an intersecting hypergraph.*

Since two edges E_i and E_j in H intersect if, and only if, the corresponding edges E'_i and E'_j in L_H intersect, we have that $\Delta_0(H) = \Delta_0(L_H)$.

Theorem 3.2. *Let H be a hypergraph with l levels and no repeated edges. Then $\Delta_0(H) \leq 2^{l-1}$.*

Proof. $\Delta_0(H) = \Delta_0(L_H)$, and H has repeated edges if, and only if, L_H does too. The result then follows from [4], Ch. 1, Theorem 4. □

Theorem 3.2 is a direct generalisation of [4], Ch. 1, Theorem 4.

Theorem 3.3. *Let H be a simple intersecting hypergraph such that $r' \leq l/2$. Then $\sum_{E \in H} \binom{l-1}{r_E - 1}^{-1} \leq 1$ and $m(H) \leq \binom{l-1}{r'-1}$.*

Proof. If H satisfies the conditions, the order of L_H is l, its rank is r', and $|E'| = r_E$ for each edge E'. Moreover, $m(H) = m(L_H)$, $\Delta_0(H) = \Delta_0(L_H)$, and H is simple if, and only if, L_H is simple, so the result follows from a theorem by Erdös, Cha-Ko, and Rado ([11]). □

Theorem 3.3 improves the bounds given by that result, although it does not apply to the same class of hypergraphs, for having no edge with more than half the levels and having no edge with more than half the vertices are different and not comparable. However, see Theorem 3.4.

Theorem 3.4. *The following statements hold for any intersecting hypergraph H:*

1. $$\sum_{E\in H,\ r_E\ \leq\ l/2} \binom{l}{r_E-1}^{-1} + \sum_{E\in H,\ r_E\ >\ l/2} \binom{l}{r_E}^{-1} \leq 1$$

2. $m(H) \leq \binom{l}{[l/2]+1}$.

Proof. If H is intersecting, then L_H is intersecting, and the result follows straightforward from the generalisation due to Greene, Katona, and Kleitman of the theorem by Erdös, Cha-Ko, and Rado mentioned above ([16]). □

Theorem 3.4 improves the bounds given by that generalisation. Theorem 3.3 is a particular case of Theorem 3.4.

Definition 3.5. *For a set $V = \{x_1, \ldots, x_n\}$, a partition $P = \{P_1, \ldots, P_l\}$ of V, and an integer $r' \leq l$, the r'-level-complete hypergraph $H^{r'}_{n,l}$ has vertex set V and its edges are the subsets of V which contain r' elements of P and intersect no other element of P.*

Remark 3.6. *If every element of P has only one vertex, then we have the r-complete hypergraph. Moreover, the level hypergraph of the r'-level-complete hypergraph $H^{r'}_{n,l}$ is $K^{r'}_l$ (for every $n \geq l$).*

Theorem 3.7. $\Delta_0(H^{r'}_{n,l}) = \binom{l-1}{r'-1}$ *if* $r' \leq l/2$, *and* $\Delta_0(H^{r'}_{n,l}) = \binom{l}{r'}$ *if* $r' > l/2$.

Proof. The result follows from Remark 3.6, since $\Delta_0(K^r_n) = \binom{n-1}{r-1}$ if $r \leq n/2$, and $\Delta_0(K^r_n) = \binom{n}{r}$ if $r > n/2$ ([4], Ch. 1). □

Theorem 3.8. *Let $H = (E_1, ..., E_m)$ be a simple intersecting hypergraph with $m(H) \geq 2$ and such that $\forall \{E, F\} \subset H,\ E \cup F \neq V(H)$. Then $m(H) \leq \frac{1}{2}\binom{l}{[l/2]}$.*

Proof. If H satisfies the hypothesis, then L_H does too. The order of L_H is l, and $m(H) = m(L_H)$. The result follows from a theorem by Schönheim ([23]). □

Theorem 3.8 is a generalisation of Schönheim's statement.

Theorem 3.9. *Let H be an intersecting r'-level-uniform hypergraph. Then $\Delta(H) \geq \left(\frac{r'}{(r')^2-r'+1}\right) m(H)$.*

Proof. Since $\Delta(H) = \Delta(L_H)$ and $m(H) = m(L_H)$, the result follows from [4], Ch. 3, Corollary 1 to Theorem 14. □

This result is better than the result it emulates, since $r' \leq r$ for every hypergraph H.

4. Edge Colourings

Definition 4.1. *Let H be a hypergraph. Its chromatic index $q(H)$ is the minimum number of colours so that there exists a colouring of the edges of H in which intersecting edges have different colours. A hypergraph H has the coloured edge property whenever $q(H) = \Delta(H)$.*

Remark 4.2. *The edges of L_H have the same structure as those of H, so $q(L_H) = q(H)$. Since $\Delta(L_H) = \Delta(H)$, then a hypergraph H has the coloured edge property if, and only if, L_H has the coloured edge property.*

Moreover, H has a good, equitable, strong, or uniform edge colouring if, and only if, L_H has such a colouring as well (cf. [4] for definitions). This makes it easier to check whether a given hypergraph has a particular kind of edge colouring or not.

Theorem 4.3. *$H^{r'}_{n,l}$ has the coloured edge property if, and only if, $\exists k \in \mathbb{N}$ such that $l = kr'$.*

Proof. This follows from Remark 3.6, Remark 4.2, and a theorem by Baranyai regarding K^r_n ([2]). □

Theorem 4.3 is a generalisation of the result by Baranyai.

Observation 4.4. *The famous Erdös-Faber-Lovasz conjecture can be stated in terms of edge-colourings in the following way: Every linear hypergraph with no loops on a set of k vertices can be (strongly) edge-coloured with k colours. In level hypergraph terms, it is equivalent to the following statement: A level-linear hypergraph with k levels and without one-level edges can be (strongly) edge-coloured with k colours.*

5. Helly Property

Definition 5.1. *A hypergraph H has the Helly property if, and only if, for every intersecting family $J \subset H$ we have $\bigcap_{E \in J} E \neq \emptyset$.*

Definition 5.2. *Consider $k \in \mathbb{N}$. A hypergraph H is k-*Helly *if, and only if, for every set $J \subset H$ the following conditions are equivalent:*

1. $(I \subset J,\ |I| \leq k) \Rightarrow \bigcap_{E \in I} E \neq \emptyset$.
2. $\bigcap_{E \in J} E \neq \emptyset$.

Having the Helly property is being 2-Helly.

Remark 5.3. *Since every edge E in H induces an edge E' in L_H, and $E \cap F \neq \emptyset$ if, and only if, $E' \cap F' \neq \emptyset$, then H is k-Helly if, and only if, L_H is k-Helly.*

Theorem 5.4. *Let $H = (E_1, \ldots, E_m)$ be a simple k-Helly hypergraph such that for every $i \in \{1, \ldots, m\}$, $r_{E_i} \geq k+1$. Then $\sum_{i=1}^{m} \binom{l-1}{r_{E_i} - 1}^{-1} \leq 1$.*

Proof. If H satisfies the conditions, then $L_H = (E'_1, \ldots, E'_m)$ is a simple k-Helly hypergraph of order l and such that for every $i \in \{1, \ldots, m\}$, $|E'_i| = r_{E_i}$. The result follows from a theorem by Tuza ([26]). □

Theorem 5.5. *Let $H = (E_1, \ldots, E_m)$ be a simple k-Helly hypergraph such that for every $i \in \{1, \ldots, m\}$, $r_{E_i} \geq k+2$. Then $m(H) \leq \binom{l-1}{r'-1}$.*

Proof. Similar to that of Theorem 5.4. The result follows from a theorem by Bollobás and Duchet ([8]). □

Theorem 5.6. *Let $H = (E_1, \ldots, E_m)$ be a simple hypergraph with the Helly property such that $3 \leq r' \leq l/2$. Then $m(H) \leq \binom{l-1}{r'-1}$.*

Proof. The result follows from a theorem by Bollobás and Duchet ([9]). □

Theorem 5.7. *Let $H = (E_1, \ldots, E_m)$ be a simple hypergraph with the Helly property such that $5 \leq l$. Then $m(H) \leq \binom{l-1}{[l/2]}$.*

Proof. The result follows from a theorem by Bollobás and Duchet ([9]). □

Theorems 5.4, 5.5, 5.6 and 5.7 give better bounds than the results in which they are based, but their conditions are stronger. These results (and many others) show that the better bound is attained by a level hypergraph.

6. Sections

Definition 6.1. *For a simple hypergraph $H = (E_1, \ldots, E_m)$ and $k \in \mathbb{N}$, $k \leq r$, the* k-section *of H is a hypergraph $[H]_k$ with $V([H]_k) = V(H)$ and whose edges are the sets $F \subset V(H)$ satisfying one of the following conditions:*

1. *$|F| = k$ and $F \subset E_i$ for some $i \in \{1, \ldots, m\}$,*
2. *$|F| < k$ and $F = E_i$ for some $i \in \{1, \ldots, m\}$.*

In general, the relation between k-sections and level hypergraphs is not strong, since the former only consider if two given vertices belong to the same edge, so that vertices in the same level of a hypergraph H may be "separated" by taking k-sections of H.

The k-section of a level hypergraph L_H is a level hypergraph with the same level structure as L_H (both have the same number of levels, and two levels are adjacent in L_H if, and only if, they are adjacent in its k-section) although it may have more edges, as shown in Figure 1. Since the 2-section of any hypergraph H is a graph, then $[H]_2$ is a level hypergraph unless $[H]_2$ has connected components isomorphic to K_2 (with multiple edges, see [14]), that is, unless H has connected components isomorphic to K_2 (with multiple edges).

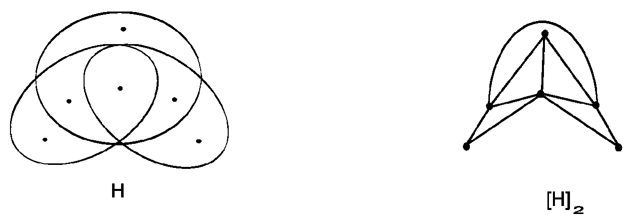


Figure 1

Definition 6.2. *A hypergraph H is conformal if, and only if, the maximal cliques of $[H]_2$ are edges of H.*

Proposition 6.3. *A hypergraph H is conformal if, and only if, L_H is conformal.*

Proof. Given a hypergraph H, its edges are always cliques of $[H]_2$. Given two vertices x and y in the same level P_i of H, (x, y) is an edge of $[H]_2$ and $N(x) \setminus \{y\} = N(y) \setminus \{x\}$.

Let H be a conformal hypergraph. We will prove by induction that L_H is conformal. If only one level P_i of H has more than one vertex, we consider two vertices x and y in P_i. Let $(F_1, \ldots, F_k)$ be the family of edges containing P_i, which are all maximal cliques, since H is conformal. Taking y away, we have the hypergraph $H[V \setminus \{y\}]$, in which the edges containing P_i are $F_1 \setminus \{y\}, \ldots, F_k \setminus \{y\}$, and other edges are not changed. Suppose some edge $F_j \setminus \{y\}$ is not a maximal clique of $[H[V \setminus \{y\}]]_2$, and consider the maximal clique Q to which $F_j \setminus \{y\}$ belongs. In $[H[V \setminus \{y\}]]_2$ there is an edge between any two vertices of Q, then it is so in $[H]_2$ too. Since x, belonging to P_i, is a vertex of Q, then y is incident with all vertices of Q in $[H]_2$, which implies that F_j is not a maximal clique in $[H]_2$, so that H is not conformal. Then, we may take away all vertices but one from P_i and the resulting hypergraph remains conformal.

Now suppose that for every conformal hypergraph in which at most $p-1$ levels have more than one vertex, its level hypergraph is conformal. Let H be a conformal hypergraph with p levels having more than one vertex. We may proceed as in last paragraph, obtaining a conformal hypergraph with only $p - 1$ levels with more than one vertex whose level hypergraph is that of H.

Let H be a hypergraph such that L_H is conformal. Following a reasoning similar to that of last two paragraphs, we may add vertices one by one (since there is a finite number of them) without altering conformality, thus showing that H is conformal. □

This result makes it easier to check if a given hypergraph is conformal.

Definition 6.4. *For every hypergraph H and every edge E of H with $|E| \geq k$, the edges of $[H]_k$ contained in E constitute a k-complete hypergraph. H is k-conformal if, and only if, the maximal subsets of $V(H)$ which constitute k-complete hypergraphs of $[H]_k$ are edges of H.*

Proposition 6.5. *A hypergraph H is k-conformal if, and only if, L_H is k-conformal.*

Proof. L_H^* is like H^* without repeated edges [14], and being or not k-Helly is not altered by adding or removing repeated edges. Then the result follows because a hypergraph is k-conformal if, and only if, its dual is k-Helly ([4], Ch. 1). □

Proposition 6.3 is a particular case of Proposition 6.5.

Definition 6.6. *Given a hypergraph $H = (E_1, \ldots, E_m)$ on a set V, its representative graph $R(H)$ is that whose vertices are the edges of H and whose edges are $\{(E_i, E_j)|\ E_i \cap E_j \neq \emptyset$ in $H\}$.*

In other words, $R(H) = [H^*]_2$.

Proposition 6.7. $R(H) = R(L_H)$.

Proof. Since L_H^* is like H^* without repeated edges, then $[L_H^*]_2 = [H^*]_2$, for adjacency is not altered by removing or adding repeated edges. □

Definition 6.8. *Let $H = (E_1, \ldots, E_m)$ be a hypergraph and let $a \geq 0$ be an integer. Multiplying the edge E_i by a means to replace E_i by a identical copies of it. For an edge E, we will write aE for the result of multiplying E by a.*

Definition 6.9. *Let $H = (E_1, \ldots, E_m)$ be a hypergraph. H is regular if all its vertices have the same degree. H is regularisable if a regular hypergraph may be obtained by multiplying each of its edges E_i by an integer $a_i > 0$.*

Theorem 6.10. *Let H be an r'-level-uniform hypergraph without vertices of degree* 1, *and such that all of its edges meet at least r' other edges. Then $R(H)$ is regularisable.*

Proof. Theorem 6.10 is a consequence of Proposition 6.7 and a theorem by Berge ([5]). □

Definition 6.11. *Given a graph G, we denote by $\omega(G)$ the intersection number of G, that is, the minimum order of those hypergraphs H whose representative graph is G.*

Given a graph G, it is easily seen that every hypergraph whose representative graph is G and whose order is $\omega(G)$ is a level hypergraph.

7. Transversals and Matchings

Proposition 7.1. *For any hypergraph H we have that: $\tau(H) = \tau(L_H)$, $\tau'(H) = \tau'(L_H)$, $\nu(H) = \nu(L_H)$, $\alpha(H) = \alpha(L_H)$, and $\rho(H) = \rho(L_H)$.*

Proof. Since a covering of H, as well as one of L_H, must cover all levels of P, and since given two edges E and F in H, $E \cap F \neq \emptyset \Leftrightarrow E' \cap F' \neq \emptyset$, we have that $\rho(H) = \rho(L_H)$. The proof of the other statements is equally straightforward and appears in [14]. □

Definition 7.2. *Given a hypergraph H, the hypergraph $Tr(H)$ has vertex set $V(H)$ and its edges are the minimal transversals of H.*

Remark 7.3. *Although for every minimal transversal of H there is a minimal transversal of L_H with vertices in the same levels (see [14]), H may have a larger number of minimal transversals, which are distinct edges of $Tr(H)$. However, for every edge of $Tr(H)$ there is an edge of $Tr(L_H)$ with the same cardinality. In particular, rank and anti-rank are equal for both hypergraphs, and $Tr(H)$ is uniform if, and only if, $Tr(L_H)$ is uniform. $Tr(L_H)$ is not always a level hypergraph, as shown in Figure 2. However, if H is a hypergraph without loops, then $Tr(H)$ is a level hypergraph.*

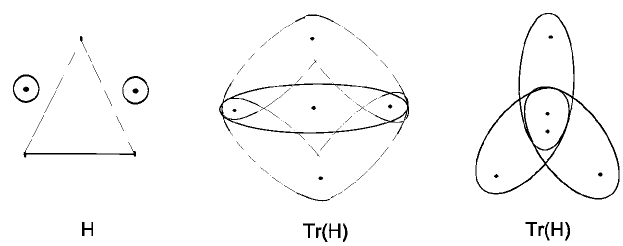


Figure 2

Theorem 7.4. *Let $H = (E_1, \ldots, E_m)$ be a hypergraph on a set V with natural partition P, and take $t = \tau'(H)$. Consider $k \in \mathbb{N}$ such that $k < s'$ and such that every k-tuple of P is contained in at most λ edges of H. Then $\sum_{i=1}^{t} \binom{r_{E_i} - 1}{k} \leq \lambda \binom{l - t}{k}$.*

Proof. Since $\tau'(H) = \tau'(L_H)$, the result follows from [4], Ch. 2, Theorem 3. □

The bound given by Theorem 7.4 is better than that appearing on [4], but it applies to a tighter class of hypergraphs: $k < r_E$ implies $k < |E|$, and every k-tuple of V is contained in at most λ edges of H if, and only if, every k-tuple of P is contained in at most λ edges of H.

Definition 7.5. *Consider a hypergraph* $H = (E_1, \ldots, E_m)$ *on a set* V. *The* complement *of* H *is* $\overline{H} = (V \setminus E_1, \ldots, V \setminus E_m)$.

It can be easily verified that $\overline{L_H} = L_{\overline{H}}$, and that a hypergraph H is a level hypergraph if, and only if, $\overline{H}$ is a level hypergraph.

Theorem 7.6. *Let* $H = (E_1, \ldots, E_m)$ *be a hypergraph on a set* V. *Then* $\tau'(H) \leq k$ *if, and only if, the hypergraph* $\overline{L_H}$ *is* k*-conformal.*

Proof. The result follows from a theorem by Berge and Duchet ([6]) and Proposition 6.5, since $\tau'(H) = \tau'(L_H)$ and $\overline{L_H} = L_{\overline{H}}$. □

Theorem 7.6 improves the result by Berge and Duchet, for it is easier to check k-conformality in $\overline{L_H} = L_{\overline{H}}$ than in $\overline{H}$.

Theorem 7.7. *Let* H *be a simple hypergraph, and consider* $k \in \mathbb{N},\ k \geq 2$. *Then* $\tau'(H) \leq k$ *if, and only if, for every partial hypergraph* $\widetilde{H} \subset L_H$ *with* $|\widetilde{H}| = k+1$ *there exists an edge* $E' \in L_H$ *such that* $E' \subset \{x \in V(L_H) \mid d_{\widetilde{H}}(x) > 1\}$.

Proof. H is simple if, and only if, L_H is simple, and $\tau'(H) = \tau'(L_H)$. The result then follows from [4], Ch. 2, Corollary 1 to Theorem 5. □

It is easier to check the condition for partial hypergraphs of L_H than for partial hypergraphs of H.

Theorem 7.8. *Let* H *be a simple hypergraph with* $\tau(H) = t \geq 2$. *Then* $Tr(H)$ *is uniform if, and only if, for every partial hypergraph* $\widetilde{H} \subset L_H$ *with* $|\widetilde{H}| = t+1$ *there exists an edge* $E' \in L_H$ *such that* $E' \subset \{x \in V(L_H) \mid d_{\widetilde{H}}(x) > 1\}$.

Proof. Since $\tau(H) = \tau(L_H)$, the result follows from Remark 7.3 and [4], Ch. 2, Corollary 2 to Theorem 5. □

As in Theorem 7.7, it is easier to check the condition for partial hypergraphs of L_H than for partial hypergraphs of H.

Definition 7.9. *A hypergraph* $H = (E_1, \ldots, E_m)$ *is* τ-critical *if, and only if,* $\forall E_i \in H,\ \tau(H - E_i) < \tau(H)$.

Proposition 7.10. *A hypergraph* H *is* τ*-critical if, and only if,* L_H *is* τ*-critical.*

Proof. Every minimal transversal (and therefore every minimum transversal) of H has at most one vertex in any given level (see [14]). Let $H = (E_1, \ldots, E_m)$ be a τ-critical hypergraph and take $E_i \in H$. There is a minimum transversal T of $H - E_i$ with $\tau - 1$ vertices. By taking $T \cap P_i$ as the representative of P_i for every level of H such that

$T \cap P_i \neq \emptyset$, we have that $T \subset V(L_H)$ meets all edges of L_H but E_i', the edge induced in L_H by E_i, since it meets all edges of H but E_i. Since $\tau(H) = \tau(L_H)$, we conclude that L_H is τ-critical.

Now consider a hypergraph $H = (E_1, \ldots, E_m)$ such that L_H is τ-critical. For every edge E_i, there is a transversal T_i of $L_H - E_i'$ with $\tau - 1$ vertices. Since T_i meets every edge of H but E_i and $\tau(H) = \tau(L_H)$, then H is τ-critical. □

Observation 7.11. *At a first glance it may seem that given a hypergraph H and an edge $E \in H$, if E' is the edge induced in L_H by E, then $L_{H-E} = L_H - E'$. It is not so, and $L_H - E'$ is not always a level hypergraph, as shown on Figure 3.*

Figure 3

Theorem 7.12. *Let $H = (E_1, \ldots, E_m)$ be a τ-critical hypergraph with $\tau = t + 1$. Then* $\sum_{i=1}^{m} \binom{r_{E_i} + t}{t}^{-1} \leq 1.$

Proof. Since $\tau(H) = \tau(L_H)$, the result follows from Proposition 7.10 and a theorem by Tuza ([26]). □

The bound given by Theorem 7.12 is better than that given by Tuza.

Theorem 7.13. *Let $H = (E_1, \ldots, E_m)$ be a τ-critical hypergraph with $\tau = t + 1$ and such that no edge has more than r' levels. Then $m(H) \leq \binom{r' + t}{t}$.*

Proof. Since $m(H) = m(L_H)$, the proof follows from a theorem by Bollobás and (independently) by Jaeger and Payan ([19]). □

The bound given by Theorem 7.13 is better than the bound given by the result mentioned, and it applies to a wider class of hypergraphs, since having no more than a vertices implies having no more than a levels.

Theorem 7.14. *Let $H = (E_1, \ldots, E_m)$ be a τ-critical hypergraph with $\tau = t + 1$. Let $\overline{A} = \{A \subset V(L_H) \mid A \notin L_H,\ \exists x \in V(L_H)$ such that $A \cup \{x\} \in L_H\}$. For $x \in V(L_H)$, $\Gamma x = \{A \in \overline{A} \mid A \cup \{x\} \in L_H\}$, and for $Y \subset V(L_H)$, $\Gamma Y = \bigcup_{x \in Y} \Gamma x$. Let $S \subset V(H)$ be a strongly stable set in H. Then $|\Gamma S'| \geq |S|$, where S' is the strongly stable set of L_H induced by S.*

Proof. A strongly stable set S in H is a strongly stable set in L_H, if we take $S \cap P_i$ as the representative of P_i for every level of H such that $S \cap P_i \neq \emptyset$. The result then follows from [4], Ch. 2,Theorem 7. □

Since an edge of L_H has at most as many vertices as its corresponding edge of H, Theorem 7.14 gives a better bound than the result in which it is based.

Theorem 7.15. *Let $H = (E_1, \ldots, E_m)$ be an intersecting level-linear hypergraph without repeated edges. Then $m \leq l$.*

Proof. Since $m(H) = m(L_H)$ and the order of L_H is l, the result follows from a theorem by DeBrujin and Erdös, completed by Ryser ([22]). □

Theorem 7.15 generalises the result mentioned, since $l \leq n$, and asking the intersection of two edges to be exactly one level is weaker than asking it to be exactly one vertex. The cases (all already known) in which $|E_i| \cap |E_j| = 1$ and $m = n$ are, of course, level hypergraphs. [4], Ch. 2, Theorem 8, which is a stronger result by Seymour, was generalised in [14].

Theorem 7.16. *Let H be an r'-level-uniform hypergraph with $\Delta = 2$. Then $\tau(H) \leq \left[\left[\frac{2l}{r'}\right]\frac{2}{3}\right]$ if r' is even, and $\tau(H) \leq \left[\frac{4l}{3r'+1}\right]$ if r' is odd.*

Proof. The result follows from a theorem by Sterboul ([25]), since $\tau(H) = \tau(L_H)$ and $\Delta(H) = \Delta(L_H)$. □

The bound given by Theorem 7.16 is better than Sterboul's, since $r - r' \leq n - l$ for every hypergraph H. However, it applies to a different class of hypergraphs.

Theorem 7.17. *Let H be a 3-level-uniform regular hypergraph with $\Delta = 3$. Then $\tau(H) \leq \left[\frac{l}{2}\right]$.*

Proof. The result follows directly from a theorem by Henderson and Dean ([18]). □

Theorem 7.18. *Let H be an r'-level-uniform hypergraph which is level-linear and regularisable, and such that none of the partial subhypergraphs of L_H is a projective plane of order r. Then $\nu(H) \geq \frac{m}{l-1}$.*

Proof. The statement follows from [4], Ch. 3, Exercise 9. □

Theorems 7.17 and 7.18 give better bounds than the original results, but they do not apply to the same classes of hypergraphs.

8. König Property

Definition 8.1. *A hypergraph H has the König property if, and only if, $\tau(H) = \nu(H)$.*

Definition 8.2. *A hypergraph H has the dual König property if, and only if, $\rho(H) = \alpha(H)$.*

Proposition 8.3. *Let H be a hypergraph. Then H has the König property if, and only if, L_H has the König property, and H has the dual König property if, and only if, L_H has the dual König property.*

Proof. The result follows straightforwardly from Proposition 7.1. □

Theorem 8.4. *A hypergraph H with the König property has a set of k levels meeting all edges if, and only if, $k\Delta(\widetilde{H}) \geq m(\widetilde{H})$ for every $\widetilde{H} \subset L_H$.*

Proof. The result follows from Proposition 8.3 and [4], Ch. 3, Corollary 2 to Theorem 1, since for every vertex x in $V(H)$ there is a vertex y in $V(L_H)$ such that x and y belong to the same level, that is, the edges of $H(y)$ are those induced by the edges of $H(x)$. □

It is easier to check the condition in L_H than in H, and if H has k levels meeting all edges then it has k vertices meeting all edges.

Theorem 8.5. *A hypergraph H with the König property has k disjoint edges if, and only if, for every $A \subset V(L_H)$, $ks(L_H[A]) \geq |A|$.*

Proof. The result follows from Proposition 8.3 and [4], Ch. 3, Corollary 1 to Theorem 1, since two edges are disjoint in H if, and only if, their induced edges are disjoint in L_H. □

It is easier to check the condition in L_H than in H.

Theorem 8.6. *Let H be a hypergraph with the dual König property. $V(H)$ may be covered with k edges if, and only if, for every $A \subset V(L_H)$, $kr(L_H[A]) \geq |A|$.*

Proof. The result follows from Proposition 8.3 and [4], Ch. 3, Corollary to Theorem 1', since a family of edges cover H if, and only if, their induced edges cover L_H. □

As in Theorem 8.5, it is easier to work with L_H than with H.

9. Fractional transversals and fractional matchings

Definition 9.1. *Given a hypergraph H and $k \in \mathbb{N}$, a k-matching of H is a function $q : H \to \{0, 1, \dots, k\}$ such that for each vertex $x \in V$, $\sum_{E \in H(x)} q(E) \leq k$. The value of a k-matching is $\sum_{E \in H} q(E)$, and $\nu_k(H)$ is the maximum value of the k-matchings in H.*

Definition 9.2. *Given a hypergraph H, a fractional matching of H is a function $q : H \to [0,1]$ such that for each vertex $x \in V$, $\sum_{E \in H(x)} q(E) \le 1$. The value of a fractional matching is $\sum_{E \in H} q(E)$, and $\nu^*(H)$ is the maximum value of the fractional matchings in H.*

Definition 9.3. *Given a hypergraph H and $k \in \mathbb{N}$, $k \ge 1$, a k-transversal of H is a function $p : V(H) \to \{0, 1, \dots, k\}$ such that for each edge $E \in H$, $\sum_{x \in E} p(x) \ge k$. The value of a k-transversal is $\sum_{x \in V(H)} p(x)$, and $\tau_k(H)$ is the minimum value of the k-transversals in H.*

Definition 9.4. *Given a hypergraph H, a fractional transversal of H is a function $p : V(H) \to [0,1]$ such that for each edge $E \in H$, $\sum_{x \in E} p(x) \ge 1$. The value of a fractional transversal is $\sum_{x \in V(H)} p(x)$, and $\tau^*(H)$ is the minimum value of the fractional transversals in H.*

Definition 9.5. *Given a hypergraph H and $k \in \mathbb{N}$, a k-covering of H is a function $q : H \to \{0, 1, \dots, k\}$ such that for each vertex $x \in V$, $\sum_{E \in H(x)} q(E) \ge k$. The value of a k-covering is $\sum_{E \in H} q(E)$, and $\rho_k(H)$ is the minimum value of the k-coverings in H.*

Proposition 9.6. $\nu_k(H) = \nu_k(L_H)$, $\nu^*(H) = \nu^*(L_H)$, $\tau_k(H) = \tau_k(L_H)$, $\tau^*(H) = \tau^*(L_H)$, *and* $\rho_k(H) = \rho_k(L_H)$.

Proof. A k-matching, a fractional matching, and a k-covering of H are trivially a k-matching, a fractional matching, and a k-covering of L_H, and vice versa, so that $\nu_k(H) = \nu_k(L_H)$, $\nu^*(H) = \nu^*(L_H)$, and $\rho_k(H) = \rho_k(L_H)$. A proof of $\tau^*(H) = \tau^*(L_H)$ appears in [14].

$\tau_k(H) = \tau_k(L_H)$: Consider a k-transversal p of H whose value is $\tau_k(H)$. If there were a level P_i of H such that $\sum_{x \in P_i} p(x) > k$, then we could take a vertex $y \in P_i$ such that $p(y) > 0$ and define $\overline{p} : V(H) \to \{0, 1, \dots, k\}$, $\overline{p}(x) = p(x)$ if $x \ne y$, $\overline{p}(y) = p(y) - 1$, which is a k-transversal of H whose value is less than that of p. Therefore, for any k-transversal p of H whose value is $\tau_k(H)$, and for every level P_i of H, we have that $\sum_{x \in P_i} p(x) \le k$. Then the function $p' : V(L_H) \to \{0, 1, \dots, k\}$, $p'(x_i) = \sum_{x \in P_i} p(x)$ is a k-transversal of L_H whose value is that of p, that is, $\tau_k(H) \le \tau_k(L_H)$.

Now take a k-transversal p' of L_H and consider the function $p : V(L_H) \to \{0, 1, \dots, k\}$, $p(x_i) = p'(x_i)$, $p(x) = 0$ for every $x \in V(H)$ which is not the representative of any level. Then p is a k-transversal of H whose value is that of p', which implies $\tau_k(H) \ge \tau_k(L_H)$. $\square$

Theorem 9.7. *Let H be an r'-level-uniform regular hypergraph. Then $p : V(H) \to [0,1]$ such that $\forall x \in P_i$, $p(x) = \frac{1}{|P_i|\ r'}$ is an optimal fractional transversal of H.*

Proof. If H satisfies the conditions of Theorem 9.7, then L_H is a regular r'-uniform hypergraph, so $q : V(H) \to [0,1]$ such that $\forall x \in V(H)$, $q(x) = \frac{1}{r'}$ is an optimal fractional transversal of L_H ([4], Ch. 3, Corollary 3 to Theorem 1). To get a fractional transversal of H, we divide the value of each vertex in $V(L_H)$ between all vertices on the level it represents. The fractional transversal obtained in this way is optimal because $\tau^*(H) = \tau^*(L_H)$. □

Definition 9.8. *Let $H = (E_1, \ldots, E_m)$ be a hypergraph. H is quasi-regularisable if a regular hypergraph may be obtained by multiplying each of its edges E_i by an integer $a_i \geq 0$.*

Theorem 9.9. *Let H be an r'-level-uniform hypergraph. Then H is quasi-regularisable if, and only if, $\tau^*(H) = \frac{l}{r'}$.*

Proof. The result follows from [4], Ch. 3, Theorem 7, because a hypergraph H is quasi-regularisable if, and only if, L_H is quasi-regularisable [14]. □

Theorem 9.9 implies that for every quasi-regularisable hypergraph which is both uniform and level-uniform, we have $\frac{n}{r} = \frac{l}{r'}$.

Theorem 9.10. *Let H be an r'-level-uniform hypergraph with $r' \geq 3$, and such that L_H contains no projective plane of rank r as a partial subhypergraph. Then $\tau^*(H) \leq (r'-1)\nu(H)$.*

Proof. It is a consequence of [4], Ch. 3, Corollary 3 to Theorem 14. □

[4], Ch. 3, Theorem 14, a result by Füredi ([13]), as well as the other corollaries to it appearing in [4], are emulated in [14]. Theorems 9.9 and 9.10 give better bounds than the results in which they are based, but do not apply to the same classes of hypergraphs.

10. Product Hypergraphs

Definition 10.1. *Given a hypergraph $H = (E_1, \ldots, E_m)$ and a hypergraph $H' = (F_1, ..., F_{m'})$, their product $H \times H'$ is the hypergraph with $V(H \times H') = V(H) \times V(H')$ and whose edges are the sets $E_i \times F_j$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m'\}$.*

Proposition 10.2. *Given two hypergraphs H and H', we have $l(H \times H') = l(H) \cdot l(H')$, $r'(H \times H') = r'(H) \cdot r'(H')$, $s'(H \times H') = s'(H) \cdot s'(H')$, $L_{H \times H'} = L_H \times L_{H'}$, and $H \times H'$ is a level hypergraph if, and only if, H and H' are so too.*

Proof. $l(H \times H') = l(H) \cdot l(H')$: If x_1 and x_2 belong to different levels of H, then there is an edge $E \in H$ such that $x_1 \in E$ and $x_2 \notin E$, or $x_2 \in E$ and $x_1 \notin E$. Then all vertices $(x_1, y),\ y \in V(H')$ belong to a different level of $H \times H'$ than all vertices $(x_2, y),\ y \in V(H')$. The same happens with two vertices y_1 and y_2 which belong to different levels of H'. Then $l(H \times H') \geq l(H) \cdot l(H')$.

If two vertices (x_1, y_1) and (x_2, y_2) belong to different levels of $H \times H'$, then there is an edge $E_i \times F_j$ such that $(x_1, y_1) \in E_i \times F_j$ and $(x_2, y_2) \notin E_i \times F_j$, or $(x_1, y_1) \notin E_i \times F_j$ and $(x_2, y_2) \in E_i \times F_j$. Suppose the former holds, then $x_1 \in E_i$, $y_1 \in F_j$, and either $x_2 \notin E_i$ or $y_2 \notin F_j$, that is, either x_1 and x_2 belong to different levels of H, or y_1 and y_2 belong to different levels of H', which implies $l(H \times H') \leq l(H) \cdot l(H')$.

$r'(H \times H') = r'(H) \cdot r'(H')$ and $s'(H \times H') = s'(H) \cdot s'(H')$: From the above reasoning, $r_{E_i \times F_j} = r_{E_i} \cdot r_{F_j}$. The results follow straightforwardly.

$L_{H \times H'} = L_H \times L_{H'}$: Consider two hypergraphs H and H', a level P_i of H and a level Q_j of H'. Then $P_i \times Q_j = \{(x, y) \,|\, x \in P_i,\ y \in Q_j\}$ is contained in every edge $E \times F$ such that $P_i \subset E$, $Q_j \subset F$. On the other hand, $P_i \times Q_j$ is not contained in any edge $E \times F$ such that $P_i \not\subseteq E$, nor in any edge $E \times F$ such that $Q_j \not\subseteq F$. This means that the levels of $H \times H'$ are the sets $P \times Q$ such that P is a level of H and Q is a level of H', so it does not matter if we identify all vertices of a given level $P \times Q$, or if we identify the vertices of P and those of Q and then take the product.

$H \times H'$ is a level hypergraph if, and only if, H and H' are level hypergraphs: From last paragraph, a level $P \times Q$ has only one vertex if, and only if, both levels P and Q have exactly one vertex each. □

Theorem 10.3. *Every hypergraph H satisfies $\tau^*(H) = \min_{\widetilde{H}} \dfrac{\tau(L_H \times L_{\widetilde{H}})}{\tau(L_{\widetilde{H}})}$, where $\widetilde{H}$ is a partial hypergraph of H.*

Proof. Straightforward from Proposition 9.6, Proposition 10.2, and [4], Ch. 3, Theorem 16. □

It is easier to work with L_H than with H.

Theorem 10.4. *Every hypergraph H with the Helly property satisfies $\tau^*(H) = \max_{\widetilde{H}} \dfrac{\nu(L_H \times L_{\widetilde{H}})}{\nu(L_{\widetilde{H}})}$, where $\widetilde{H}$ is a partial hypergraph of H.*

Proof. The result follows from Remark 5.3, Proposition 9.6, Proposition 10.2, and [4], Ch. 3, Theorem 17. □

11. Vertex Colourings

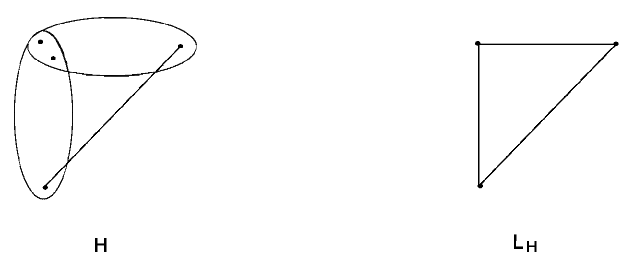
Vertex colourings are studied in [14]. Here appear some results regarding uniform colourings, the Kneser number, and the cochromatic number.

Definition 11.1. *Let H be a hypergraph on a set V, and let $|V| = n$. A k-colouring $(S_1,\ldots,S_k)$ of H is uniform if, and only if, $\left[\frac{n}{k}\right] \leq |S_i| \leq \left[\frac{n}{k}\right]^*$ for every $i \in \{1,\ldots,k\}$.*

Proposition 11.2. *If L_H has a uniform k-colouring, then H does too.*

Proof. Suppose L_H has a uniform k-colouring $S = (S_1,\ldots,S_k)$. To get a uniform k-colouring of H, take S and re-label so that $|S_i| \geq |S_{i+1}|$ for every $i \in \{1,\ldots,k-1\}$. Consider an uncoloured vertex x and colour it with S_{j+1}, the colour class such that $|S_j| > |S_{j+1}|$, if it exists; if there is not such a class, colour x with S_1. Keep on colouring uncoloured vertices in order, that is, S_{i+1} after S_i, and S_1 after S_k. It does not matter which (uncoloured) vertex gets which colour. The result is a k-coloration S' of H, since we started with a k-coloration of L_H, so there are no monochromatic edges. Since S is uniform and because of the way in which the vertices of $V(H) \setminus V(L_H)$ were coloured, S' is uniform. □

The converse is not true, as shown in Figure 4, where H has a uniform 2-colouring and L_H does not even have a 2-colouring.



Fiigure 4

Theorem 11.3. *Every hypergraph H such that $\sum_{E\in H} 2^{-r_E} + \max_x \sum_{E\in H(x)} 2^{-r_E} < 1$ admits a uniform 2-colouring.*

Proof. Straightforward from Proposition 11.2 and [4], Ch. 4, Theorem 4. □

Theorem 11.3 is a generalisation of the result in which it is based.

Theorem 11.4. *Every hypergraph H with $s' > 1$ and such that $\sum_{E\in H} \binom{l-r_E}{[l/2]} < \binom{l-1}{[l/2]}$ admits a uniform 2-colouring.*

Proof. The result follows directly from Proposition 11.2 and [4], Ch. 4, Theorem 5. □

Theorem 11.4 applies to a different class of hypergraphs than [4], Ch. 4, Theorem 5.

Theorem 11.5. *Let H be a hypergraph. Consider $k \le l$, $p = \left[\frac{l}{k}\right]$, $q = l - pk$. If $k \sum_{E\in H} \binom{l-r_E}{l-p} + q \sum_{E\in H} \binom{r_E}{p+1-r_E} < \binom{l}{p}$, then H admits a uniform k-colouring.*

Proof. It is a consequence of Proposition 11.2 and a generalisation due to Hansen and Loréa ([17]) to [4], Ch. 4, Theorem 5. □

Definition 11.6. *Given a hypergraph H, its Kneser number $\tau_0(H)$ is the minimum number of intersecting families whose union is H. Its k-Kneser number $\kappa_k(H)$ is the minimum number of intersecting families needed to cover k times H. Its fractional Kneser number is $\tau_0^*(H) = \min_{k\ge 1}\left(\frac{\kappa_k(H)}{k}\right)$.*

Proposition 11.7. *For any hypergraph H, the following statements hold: $\tau_0(H) = \tau_0(L_H)$, $\kappa_k(H) = \kappa_k(L_H)$, $\tau_0^*(H) = \tau_0^*(L_H)$.*

Proof. Straightforward, since two edges intersect in H if, and only if, their induced edges intersect in L_H, and a set of edges covers k times H if, and only if, the set of its induced edges covers k times L_H. □

Theorem 11.8. *Let H be an r'-level-uniform hypergraph with $l \ge 2r'$. Then $\tau_0(H) \le l - 2r' + 2$.*

Proof. The result follows from Proposition 11.7 and [4], Ch. 4, Proposition. □

The bound given by Theorem 11.8 is better than that of [4], Ch. 4, Proposition, but it applies to a different class of hypergraphs.

Definition 11.9. *Given a hypergraph H, its cochromatic number $\overline{\gamma}(H)$ is the smallest positive integer k such that for every k-partition $S = (S_1, \ldots, S_k)$ of $V(H)$ there is an edge $E \in H$ such that $\{x, y\} \in E,\ x \in S_i,\ y \in S_j \Rightarrow i \ne j$.*

Proposition 11.10. $\overline{\gamma}(L_H) \le \overline{\gamma}(H)$.

Proof. Let $\overline{\gamma}(H) = k$ and take a k-partition $S = (S_1, \ldots, S_k)$ of $V(H)$. Let $E \in H$ be an edge such that $\{x, y\} \in E,\ x \in S_i,\ y \in S_j \Rightarrow i \ne j$. If we colour the representative of each level with any colour present on that level, we get a k-partition $S' = (S'_1, \ldots, S'_k)$ of $V(L_H)$ such that $\{x, y\} \in E',\ x \in S'_i,\ y \in S'_j \Rightarrow i \ne j$. □

It is clear that the converse is not true, since given a level hypergraph L_H, there exists a hypergraph H such that L_H is its level hypergraph and every edge $E \in H$ has as many vertices as desired.

12. Cycles

If L_H has a cycle C of length k, then C is a cycle of H too. However, for any positive integer k there are hypergraphs with cycles of length k whose level hypergraphs have no cycles at all [14].

Definition 12.1. *Let H be a hypergraph. A B-cycle $(x_1, E_1, \ldots, E_k, x_1)$ of H is an odd cycle such that no vertex of $V(H)$ is in more than two edges of it, and such that $|E_i \cap E_{i+1}| = 1$ for $i \in \{1, \ldots, k-1\}$ (although $|E_k \cap E_1| \geq 1$).*

Proposition 12.2. *If H has a B-cycle of length k then L_H has a B-cycle of length k.*

Proof. Let $C = (x_1, E_1, \ldots, E_k, x_1)$ be a B-cycle of length k in H. Since no vertex of $V(H)$ is in more than two edges of C, all vertices in the set $\{x_1, \ldots, x_k\}$ belong to different levels of H (in fact, all but x_1 belong always to single-vertex levels), so that $C' = (x_1, E'_1, \ldots, E'_k, x_1)$ is a cycle of L_H, taking x_1 as representative of its level. It is straightforward to check that C' is a B-cycle of L_H. □

The converse is not true, as shown in Figure 5.

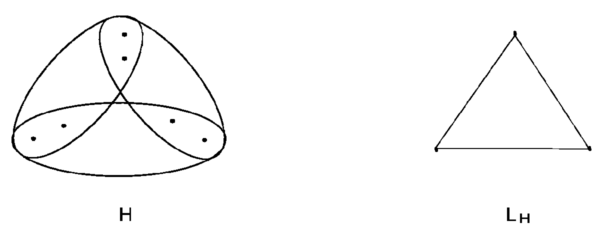


Figure 5

Definition 12.3. *Given two hypergraphs $H = (E_1, \ldots, E_m)$ and $H' = (F_1, \ldots, F_n)$ with $V(H) = V(H')$, and such that every edge of H meets at least one edge of H' and vice versa, the composition of H' with H is $H \circ H'$, whose vertices are the edges of H' and whose edges are the sets $\overline{E}_j = \{F_i \,|E_j \cap F_i \neq \emptyset\}$.*

Proposition 12.4. *$H \circ H' = L_H \circ L_{H'}$.*

Proof. For every hypergraph H, $|L_H| = |H|$, and $E_j \cap F_i \neq \emptyset$ if, and only if, $E'_j \cap F'_i \neq \emptyset$. □

Definition 12.5. *A hypergraph H is balanced if, and only if, for every odd cycle C there is an edge of C which contains at least three vertices of C.*

Theorem 12.6. *A hypergraph H is balanced if, and only if, every partial subhypergraph of L_H has the König property.*

Proof. The result follows from Proposition 8.3 and a theorem by Berge and Las Vergnas ([7]), since a hypergraph is balanced if, and only if, its level hypergraph is balanced [14]. □

Definition 12.7. *A hypergraph H is unimodular if, and only if, its incidence matrix is totally unimodular.*

Theorem 12.8. *A hypergraph H is unimodular if, and only if, every induced subhypergraph of L_H has an equitable 2-colouring.*

Proof. The statement follows from [4], Ch. 5, Theorem 4, since a hypergraph is unimodular if, and only if, its level hypergraph is unimodular [14]. □

Theorems 12.6 and 12.8 make it easier to check if a given hypergraph is balanced or unimodular.

13. Hypergraphs with two levels

In [14] we stated that a multigraph is a level hypergraph if, and only if, it has no connected components isomorphic to K_2. This allows the generalisation of some results regarding multigraphs to hypergraphs H such that L_H is a multigraph, that is, to hypergraphs such that $r' = 2$. Some examples follow:

Theorem 13.1. *Let H be a hypergraph with $r' = 2$. If L_H is not a cycle of odd length, there is a 2-edge-colouring of H such that both colors are present in every vertex x with $d_H(x) \geq 2$.*

Proof. If H satisfies the hypothesis, then L_H is a multigraph and not an odd cycle, so there exists such a 2-edge-colouring S of L_H ([10], Lemma 6.1.1), which is also a 2-edge-colouring of H. Since all vertices in a given level P_i have the same degree, S satisfies the condition for H. □

Theorem 13.2. *Let H be a hypergraph with $r' = 2$. If L_H is bipartite, then $q(H) = \Delta(H)$.*

Proof. The result follows straightforward from a classical theorem regarding graphs ([10], Theorem 6.1). □

Theorem 13.3. *Let H be a simple hypergraph with $r' = 2$. Then $\Delta(H) \leq q(H) \leq \Delta(H) + 1$.*

Proof. The result follows from a known theorem by Vizing and (independently) Gupta for simple graphs ([10], Theorem 6.2). Observe that single-level edges not contained in other edges do not alter the result. □

Theorem 13.4. *Let H be a simple hypergraph such that L_H is a bipartite graph. Then $\tau(H) = \nu(H)$.*

Proof. Since $\tau(H) = \tau(L_H)$ and $\nu(H) = \nu(L_H)$, the result follows from Hall's Theorem ([10], Theorem 5.2). Single-level edges not contained in other edges do not alter the result. □

Theorem 13.5. *Let H be a hypergraph such that every edge has exactly two levels. Then $\alpha + \tau = \nu + \rho = l$. If L_H is bipartite, then $\alpha = \rho$.*

Proof. Straightforward, since all five numbers are equal for H and L_H. The second statement follows from Theorem 13.4. □

Theorem 13.6. *Let H be a hypergraph with $r' = 2$ such that L_H has no cycles of odd length. Then H has a strongly stable transversal.*

Proof. In this case, L_H is a multigraph without cycles of odd length, which has a strongly stable transversal. This implies that H has as well a strongly stable transversal. □

Theorem 13.7. *For every simple hypergraph H with $r' = 2$, $\tau^*(H) = \frac{\nu_2(H)}{2} = \frac{\tau_2(H)}{2}$. Moreover, there exists a maximum 2-matching of L_H whose connected components are loops, pairs of parallel edges, and odd cycles.*

Proof. Since H is simple, every single-level edge becomes an isolated vertex in L_H. The result then follows from Proposition 9.6 and [4], Ch. 3, Theorem 2. □

Theorem 13.8. *For every simple hypergraph H with $r' = 2$, $\tau^*(G) = \frac{1}{2}(\nu(G) + \tau(G))$.*

Proof. Straightforward from Proposition 7.1, Proposition 9.6, and a theorem by Lovász ([20]). □

Theorem 13.9. *For every simple hypergraph H with $r' = 2$, the following conditions are equivalent:*

1. $\tau^*(H) = \tau(H)$
2. $\nu(H) = \tau(H)$.

Proof. The result follows from Proposition 7.1, Proposition 9.6, and [4], Ch. 3, Corollary to Theorem 3. □

Theorem 13.10. *Every simple hypergraph H with $r' = 2$ satisfies $\tau^*(H) \leq \frac{3}{2}\nu(H)$. Equality holds if, and only if, L_H is the union of pairwise disjoint triangles.*

Proof. It is direct from Proposition 7.1, Proposition 9.6, and a theorem by Lovász ([20]). □

Theorem 13.11. *For a hypergraph H with $r' = 2$, the following statements are equivalent:*

1. *H is quasi-regularisable.*

2. $\tau^*(H) = \frac{l}{2}$.

3. *There is a (non-trivial) partial hypergraph $\widetilde{H}$ of L_H such that the connected components of $\widetilde{H}$ are isolated vertices and odd cycles.*

4. $|N_{L_H}(S)| \geq |S|$ *for every strongly stable set S of H.*

Proof. 1) $\Leftrightarrow$ 2) follows from Theorem 9.9, since single-level edges do not alter the result.

1) $\Leftrightarrow$ 3): Since L_H is a graph, this follows from [4], Ch. 3, Theorem 8.

1) $\Leftrightarrow$ 4): Same as above, since a set S is strongly stable in H if, and only if, it is strongly stable in L_H (taking $S \cap P_i$ as the representative of P_i whenever $|S \cap P_i| \neq \emptyset$). □

Notice that $|N_{L_H}(S)| \geq |S| \Rightarrow |N_H(S)| \geq |S|$ for every strongly stable set S of H, but the converse is not true. There are hypergraphs with $r' = 2$, and such that $|N_H(S)| \geq |S|$ for every strongly stable set S of H, which are not quasi-regularisable, as shown in Figure 6.

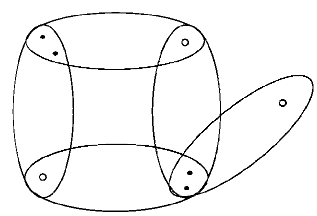


Figure 6

Theorem 13.12. *Let H be a connected hypergraph with l even and $r' = 2$, such that every two disjoint odd cycles in L_H are joined by an edge. Then H has a perfect matching if, and only if, $|N_{L_H}(S)| \geq |S|$ for every strongly stable set S of H.*

Proof. The result follows from a theorem by Fulkerson, McAndrew, and Hoffmann ([4], Ch. 3, Theorem 9), since L_H is a connected graph and H has a perfect matching if, and only if, L_H has a perfect matching. □

Theorem 13.13. *The following conditions are equivalent for any connected hypergraph H with $r' = 2$:*

1. *H is regularisable and L_H is not bipartite.*

2. $\tau^*(H) = \frac{l}{2}$, *and $t(x) = 1$ for every $x \in V(L_H)$ is the only optimal 2-transversal of L_H.*

3. $|N_{L_H}(S)| > |S|$ *for every stable set S of H.*

4. $|N_{L_H}(A)| > |A|$ *for every non-empty proper subset $A \subset V(L_H)$.*

Proof. H is regularisable if, and only if, L_H is regularisable ([14]); the result is then a consequence of Proposition 9.6 and a theorem by Berge ([5]). □

Theorem 13.14. *Let H be a connected hypergraph with $r' = 2$, and such that $K_{1,3}$ is not an induced subgraph of L_H. Then H is regularisable if, and only if, it has no vertices of degree 1 and L_H is not isomorphic to a graph consisting on an even cycle $(x_0, \ldots, x_{2p-1}, x_0)$ and a non-empty set of chords of the form (x_{2i}, x_{2i+2}).*

Proof. The result follows from a theorem by Jaeger and Payan ([21]). □

Theorem 13.15. *Every simple hypergraph H such that $r' = 2$ is k-Helly for $k \geq 3$.*

Proof. If H satisfies the hypothesis, then L_H is a simple graph but for loops in isolated vertices. The result follows from Remark 5.3 and [4], Ch. 1, Exercise 13, since isolated vertices do not affect k-Hellyness. □

Theorem 13.16. *Let $H = (E_1, \ldots, E_m)$ be a simple hypergraph with $r' = 2$ and such that $\alpha(H) = k$ and $\alpha(H - E_i) = k + 1$ for every $i \in \{1, \ldots, m\}$. Then $m \leq \binom{l-k+1}{2}$.*

Proof. Since $m(H) = m(L_H)$ and $\alpha(H) = \alpha(L_H)$, the proof follows from a theorem by Erdös, Hajnal, and Moore ([4], Ch. 2, Corollary 2 to Theorem 6). □

As stated at the beginning of this section, all results appearing in it are generalisations of the theorems in which they are based.

14. Conclusions and Scope

Level hypergraphs may be considered a tool which simplifies the research on several branches of hypergraph theory. For many questions regarding a given hypergraph H, we may restrict ourselves to its level hypergraph L_H, which is in many cases simpler and never more complex than H.

In fact, many classical results on hypergraphs were obtained thinking on level hypergraphs. In particular, several classical bounds are only attained by level hypergraphs. Then the main value of this work (along with [14]) is to show that in many cases we can consider only the simplest among a given class of hypergraphs to solve a problem regarding the whole class.

References

[1] B. D. Acharya, Unpublished Manuscript, MRI, 1979. Recently published as: B. D. Acharya, Representing hypergraphs by families of sets with given properties, *Indian J. Math.*, **51** (3) (2009), 597-609.

[2] Z. Baranyai, On the factorization of the complete uniform hypergraph, *Infinite and Finite Sets* (A. Hajnal, R. Rado, V. T. Sós, eds.), North Holland, Amsterdam 1975, 91-108.

[3] C. Berge, *The Theory of Graphs,* Dover Publications, New York 2001.

[4] C. Berge, *Hypergraphs. Combinatorics of Finite Sets,* Elsevier Science Publishers, Amsterdam 1989.

[5] C. Berge, Regularisable Graphs, Advances in Graph Theory (Bollobás, ed.), *Annals of Discrete Math.*, **3** (1978), 11-20.

[6] C. Berge and P. Duchet, Une généralisation du théorème de Gilmore, *Cahier du C.E.R.O.*, **17** (1975), 111-124.

[7] C. Berge and M. Las Vergnas, Sur un théoreme du type König pour hypergraphes, *Annals N.Y. Acad. Sc.*, **175** (1970), 32-40.

[8] B. Bollobás and P. Duchet, Helly Families of Maximal Size, *J. Comb. Theory A* , **26** (1979), 197-200.

[9]] B. Bollobás and P. Duchet, On Helly Families of Maximal Size, *J. Comb. Theory B*, **35** (1983), 290-296.

[10] J. A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Springer Verlag, 2008.

[11] P. Erdös, Cha-Ko, and R. Rado, Intersection Theorems for Systems of Finite Size, *Quart. J. Math. Oxford*, **12** (1961), 313-318.

[12] P. Erdös, A. Hajnal, and J. W. Moon, Mathematical Notes, *Amer. Math. Monthly*, **71** (1964), 1107-1110.

[13] Z. Füredi, Maximum Degrees and Fractional Matchings in Uniform Hypergraphs, *Combinatorica*, **1** (1981), 155-162.

[14] H. Galeana-Sánchez and M. Manrique, Level Hypergraphs, *AKCE Int. J. Graphs Comb.*, **6** (2009), 409-427.

[15] H. Galeana-Sánchez and M. Manrique, Directed hypergraphs: A tool for researching digraphs and hypergraphs, *Discuss. Math. Graph Theory*, **29** (2009), 313-335.

[16] C. Greene, G. O. H. Katona, and D. J. Kleitman, Extensions to the Erdös-Ko-Rado Theorem, *Recent Advances in Graph Theory,* Academia Praha 1975, 223-231.

[17] P. Hansen and M. Loréa, Deux conditions de colorabilité des hypergraphes, Coll. Math. Discrete, Bruxelles 1978, *Cahiers du C. E. R. O.* **20** (1978), p. 3-4, 405-410.

[18] J. Henderson and R. A. Dean, The 1-width of (0,1)-matrices having constant sum three, *J. Comb. Theory A*, **16** (1974), 355-370.

[19] F. Jaeger and C. Payan, Détermination du nombre d'aretes d'une hypergraphe τ-critique, *C. R. Acad. Sci. Paris*, **271** (1961), 221-223.

[20] L. Lovász, 2-matchings and 2-covers of hypergraphs, *Acta Math. Ac. Sci. Hungar*, **16** (1975), 50-57.

[21] C. Payan, Coverings by Minimal Transversals, *Discrete Math.*, **23** (1978), 273-277.

[22] H. Ryser, New Types of Combinatorial Designs, *Actes Congrès International des mathématiciens,* Nice **3** (1970), 235-239.

[23] J. Schönheim, On a Problem of Daykin Concerning Intersecting Families of Sets, *Combinatorics,* London Math. Soc. Lecture Notes Series 13, Cambridge Univ. Press (1974), 139-140.

[24] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.*, **27** (1928), 544-548.

[25] F. Sterboul, Sur le nombre transversal des hypergraphes uniformes, *Discrete Math.*, **22** (1978), 91-96.

[26] Z. Tuza, Helly-type Hypergraphs and Sperner Families, *Europ. J. Combinatorics*, **5** (1984), 185-187.