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On the planarity of iterated star-line graphs

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Abstract

Let H be connected graph with at least three vertices. The H-line graph, $HL(G)$, of a graph G has as its vertices the edges of G and two vertices of $HL(G)$ are adjacent if the corresponding edges in G are adjacent and belong to a common copy of H. If $H \cong K_{1,n}$, H-line graphs are called star-line graphs. In this paper, we obtain the conditions for the planarity of iterated star-line graphs and use it to compute the *n-star-line index*, $\zeta_n(G)$.

Keywords: Star-line graph, Star-line index, Planarity.

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1. Introduction

The line graph $L(G)$ of a graph G has as its vertices the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The triangular line graph [3], has as its vertices the edges of G and two vertices are adjacent if the corresponding edges belong to a common triangle of G. This graph is also known as the anti-Gallai graph of G, $antiGal(G)$ [4]. These concepts were generalized in [1] to obtain H-line graphs.

Let H be a connected graph of order $n \geq 3$. The H-line graph, $HL(G)$, of G has as its vertices the edges of G. Two vertices in $HL(G)$ are adjacent if the corresponding edges in G are adjacent and belong to a common copy of H. A graph G is an H-line graph if there exists graph a G' such that $G \cong HL(G')$. If $H \cong K_{1,n}, n \geq 3$, H-line graphs are called n-star-line graphs.

If $H \cong K_{1,2}$ or $H \cong K_3$, then $HL(G)$ is the line graph or the triangular line graph of G respectively. It is proved in [5] that H- line graphs admit a forbidden subgraph characterization only when $H \cong K_{1,2}$ and the star-line graphs admit a Krausz-type characterization. In [2], Ghebleh et. al studied the planarity of iterated line graphs and introduced the notion of the line index of a graph, $\zeta(G)$. In this paper, these notions are generalized to $\zeta_n(G)$, the *n-star-line index* of a graph G.

All the graphs considered here are undirected, finite and simple. For all basic concepts and notations not mentioned in this paper we refer to [6].

2. 3-star-line index of a graph

Definition 2.1. *The n-star-line index of a graph G, $\zeta_n(G)$, is the smallest k such that $K_{1,n}L^k(G)$ is nonplanar.*

If $K_{1,n}L^k(G)$ is planar for all $k \geq 0$, we define $\zeta_n(G) = \infty$.

Lemma 2.2. *If G' is a subgraph of G, then $\zeta_n(G) \leq \zeta_n(G')$.*

Lemma 2.3. *If G is a graph with $\Delta(G) \geq 4$, then $\zeta_3(G) \leq 3$, where $\Delta(G)$ is the maximum degree of G.*

Proof. If $\Delta(G) \geq 4$, then G contains $K_{1,4}$ as a subgraph and $K_{1,3}L^3(K_{1,4})$ is a 6-regular graph and hence is nonplanar. Therefore, $\zeta_3(K_{1,4}) \leq 3$ and by Lemma 2.2, $\zeta_3(G) \leq 3$. □

Lemma 2.4. *For any graph G, $\zeta_3(G) \in \{0, 1, 2, 3, 4, \infty\}$. Also, $\zeta_3(G) = \infty$ if and only if $\Delta(G) \leq 3$ and no two vertices in G of degree three are adjacent.*

Proof. If $\Delta(G) \geq 4$, by Lemma 2.3, we have $\zeta_3(G) \leq 3$. If $\Delta(G) \leq 2$, then G does not contain $K_{1,3}$ as a subgraph and hence $K_{1,3}L(G)$ is totally disconnected. Therefore $\zeta_3(G) = \infty$. If $\Delta(G) = 3$ and G does not have two adjacent vertices of degree three, then $K_{1,3}L^2(G)$ will be totally disconnected and hence $\zeta_3(G) = \infty$. If G has two adjacent vertices of degree three, then $K_{1,3}L^2(G)$ will have K_4 as a subgraph and $K_{1,3}L^2(K_4)$ is a six regular graph which is non-planar. Hence, $\zeta_3(G) \leq 4$. Also, $\zeta_3(H) = 4$, where H is the graph in Fig. 1. □

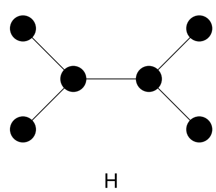

Fig. 1:

Lemma 2.5. *For any graph G, $K_{1,3}L(G)$ is planar if and only if G satisfies the following:*

(i) $\Delta(G) \leq 4$.

(ii) *G does not contain any one of the graphs H_1 or H_2 in Fig.2 as a subgraph. (An edge with a single end vertex shows the degree of that vertex.)*

(iii) *G does not contain any subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in G is at least three.*

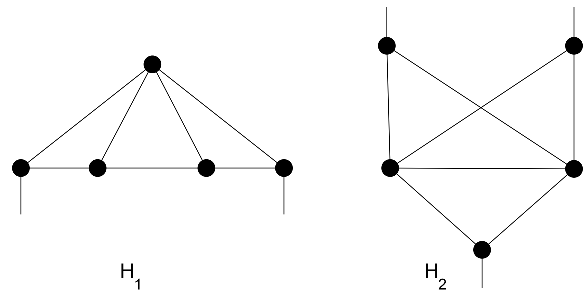

Fig. 2:

Proof. If $\Delta(G) \geq 5$, then $K_{1,3}L(G)$ contains K_5 as a subgraph and hence it is nonplanar. Also, if G has any one of the graphs H_1 or H_2 as a subgraph, then $K_{1,3}L(G)$ will contain any one of the graphs H_1' or H_2' in Fig.3 as a subgraph. H_1' and H_2' contains a homeomorphic copy of K_5 (shown in bold lines) and hence it is nonplanar. Therefore, $K_{1,3}L(G)$ is also nonplanar.

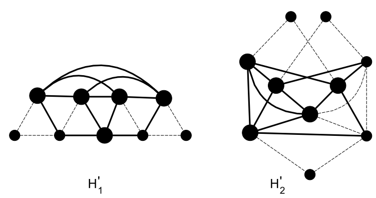

Fig. 3:

For the necessity of condition (iii) we prove the following.

Claim 1. *If G has a subgraph homeomorphic to G' in which degree of every vertex in G is at least three, then $K_{1,3}L(G)$ has a subgraph homeomorphic to $K_{1,3}L(G')$.*

Let u_1u_2 be an edge of G' and u be the vertex in $K_{1,3}L(G')$ corresponding to the edge u_1u_2. Suppose that the edge u_1u_2 is subdivided by the vertex u_3 whose degree in G is at least three as in Fig.4.

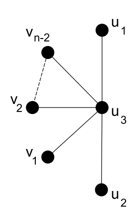

Fig. 4:

Then the edges $u_3u_1, u_3u_2, u_3v_1, u_3v_2 \ldots u_3v_{n-2}$ will form a clique C_u in $K_{1,3}L(G)$. Now, the vertices which were adjacent to u in $K_{1,3}L(G')$ will be adjacent to the vertices corresponding to u_3u_1 and u_3u_2 in $K_{1,3}L(G)$. Thus, corresponding to every edge of $K_{1,3}L(G')$, we get a path in $K_{1,3}L(G)$ and hence it contains a subgraph homeomorphic to $K_{1,3}L(G')$. Thus, Claim 1 follows.

Hence, if G has a subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in G is at least three, the $K_{1,3}L(G)$ has a subgraph homeomorphic to $K_{1,3}L(K_{3,3})$. But, $K_{1,3}L(K_{3,3})$ has a homeomorphic copy of $K_{3,3}$(shown in bold lines in Fig.5) and hence it is nonplanar.

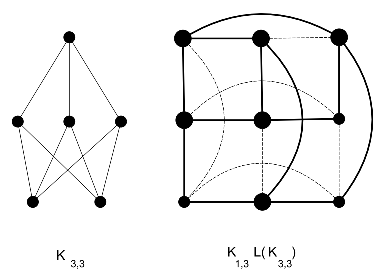

Fig. 5:

Conversely, suppose that $K_{1,3}L(G)$ is nonplanar. Then, it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

Case 1. $K_{1,3}L(G)$ *contains* K_5 *or a subgraph homeomorphic to* K_5.

If $K_{1,3}L(G)$ contains K_5, then there are five mutually incident edges in G and $\Delta(G) \geq 5$, which is a contradiction. If $K_{1,3}L(G)$ has a copy of K_5 with one edge subdivided once or twice, then it contains either a copy of G_1 or a copy of G_2 in Fig.6 as an induced subgraph. If $K_{1,3}L(G)$ has a copy of K_5 with one edge subdivided more than twice then it contains a copy of G_3 as an induced subgraph. If $K_{1,3}L(G)$ has a copy of K_5 with more than one edge subdivided, then it has a copy of $K_{1,3}$ as an induced subgraph. All the graphs $G_1, G_2, G_3, K_{1,3}$ are forbidden subgraphs for line graphs and hence are forbidden for star-line graphs also. Hence, $K_{1,3}L(G)$ cannot have any subgraph homeomorphic to K_5 other than K_5.

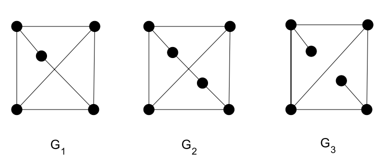

Fig. 6:

Case 2. *$K_{1,3}L(G)$ contains $K_{3,3}$ or a homeomorphic copy of $K_{3,3}$ as a subgraph.*

It is clear that any adge of a H-line graph lie in a copy of $L(H)$. In [5], it is proved that star-line graphs admit a Krausz type edge-clique partition. Any two cliques in this partition can have at most one common vertex. Also, any edge in $K_{1,3}L(G)$ will lie in a copy of $L(K_{1,3}) = K_3$. These conditions will force $K_{1,3}L(G)$ to have a copy of K_5 or a homeomorphic copy of $K_{3,3}$ in which degree of every vertex in G is at least three, whenever it contains contains $K_{3,3}$ or a homeomorphic copy of $K_{3,3}$ as a subgraph. Then, either it reduces to Case 1 or $\Delta(G)$ will be greater than four. □

Lemma 2.6. *For any graph G, $\zeta_3(G) = 1$ if and only if G is planar and contains any one of the graphs $K_{1,5}$, H_1 or H_2 in Fig.2 as a subgraph.*

Proof. By definition, $\zeta_3(G) = 1$ if and only if G is planar and $K_{1,3}L(G)$ is nonplanar. It is clear from Lemma 2.5 that $K_{1,3}L(G)$ is planar if and only if any of the following holds.

(i) $\Delta(G) \leq 4$.

(ii) G does not contain any one of the graphs H_1 or H_2 in Fig.2 as a subgraph.

(iii) G does not contain any subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in G is at least three. Since, G is to be planar condition (iii) cannot happen. Therefore, the only possibility is that G contains any one of the graphs $K_{1,5}$, H_1 or H_2 in Fig.2 as a subgraph.

□

Lemma 2.7. *For any graph G, $\zeta_3(G) = 2$ if and only if $K_{1,3}L(G)$ is planar and G contains any one of the graphs in Fig.7 as a subgraph.*

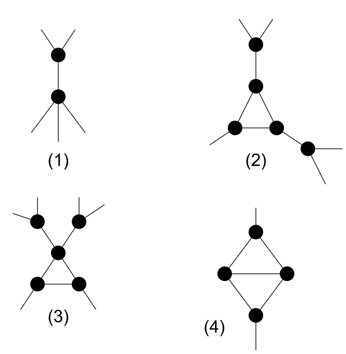

Fig. 7:

Proof. By Lemma 2.6, $\zeta_3(G) = 2$ if and only if $K_{1,3}L(G)$ has any one of the graphs $K_{1,5}$, H_1 or H_2 in Fig.2 as a subgraph. As in the proof of Lemma 2.5, it follows that this is possible if and only if G has any one of the graphs in Fig.7 as a subgraph. □

Lemma 2.8. *For any graph G, $\zeta_3(G) = 4$ if and only if $\Delta(G) \leq 3$, G is planar and has two adjacent vertices of degree three and does not have any one of the graphs in Fig.8 as a subgraph.*

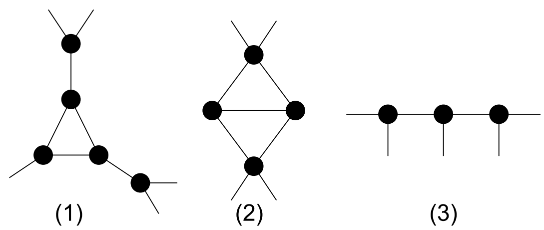

Fig. 8:

Proof. If $\Delta(G) \geq 4$, we have by Lemma 2.3 that $\zeta_3(G) \leq 3$. Also, $\zeta_3(G)$ of the graphs (1) and (2) in Fig.8 is two and that of the graph (3) in Fig.8 is three. Hence, if G contains any of these graphs as subgraphs, then by Lemma 2.2, $\zeta_3(G) \leq 3$. Now, if $\Delta(G) \leq 3$ and G does not have two adjacent vertices of degree three, then $K_{1,3}L^2(G)$ will be totally disconnected and $\zeta_3(G) = \infty$. □

We thus have,

Theorem 2.9. *Let G be any graph. Then,*

(1) $\zeta_3(G) = \infty$, *if and only if $\Delta(G) \leq 3$ and G does not contain two adjacent vertices of degree three.*

(2) $\zeta_3(G) = 0$, *if and only if G is non-planar.*

(3) $\zeta_3(G) = 1$, *if and only if G is planar and contains any one of the graphs $K_{1,5}$, H_1 or H_2 in Fig.2 as a subgraph.*

(4) $\zeta_3(G) = 2$, *if and only if $K_{1,3}L(G)$ is planar and G contains any one of the graphs in Fig.7 as a subgraph.*

(5) $\zeta_3(G) = 4$, *if and only if $\Delta(G) \leq 3$, G is planar and has two adjacent vertices of degree three and does not contain any one of the graphs in Fig.8 as a subgraph.*

(6) $\zeta_3(G) = 3$, *otherwise.* □

3. n-star-line index of a graph

We first state two lemmas which can be proved as in the previous section and use it to compute the value of $\zeta_4(G)$. We also compute $\zeta_n(G)$, for $n \geq 5$.

Lemma 3.1. *Let G be any graph. Then $K_{1,4}L(G)$ is planar if and only if G satisfies the following:*

(i) $\Delta(G) \leq 4$.

(ii) G does not contain any one of the graphs H_3 or H_4 in Fig.9 as a subgraph.

(iii) G does not contain any subgraph homeomorphic to $K_{3,3}$ in which degree of every vertex in G is at least four.

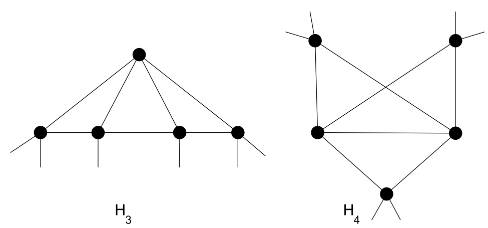

Fig. 9:

Lemma 3.2. *For any graph G, $\zeta_4(G) = 2$ if and only if $K_{1,4}L(G)$ is planar and G has any one of the graphs in Fig.10 as a subgraph.*

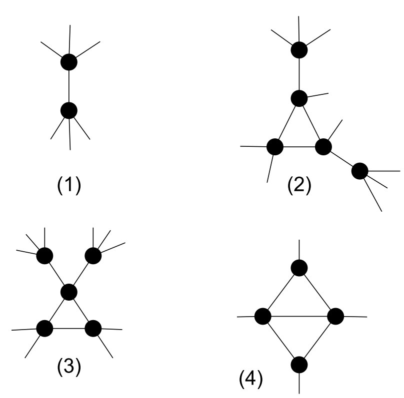

Fig. 10:

Lemma 3.3. *Let G be any graph. Then, $\zeta_4(G) = \{0, 1, 2, \infty\}$.*

Proof. For any graph G, $\zeta_4(G) = 3$ if and only if $K_{1,4}L^2(G)$ contains any one of the graphs $K_{1,5}$, H_3 or H_4 as a subgraph. Also, if $K_{1,4}L^2(G)$ contains any of these graphs, then G has any one of the graphs in Fig.10 as a subgraph, which implies that $K_{1,4}L^2(G)$ is nonplanar and $\zeta_4(G) = 2$. □

We summarize these results as follows.

Theorem 3.4. *Let G be any graph. Then,*

(1) $\zeta_4(G) = 0$, if and only if G is non-planar.

(2) $\zeta_4(G) = 1$, if and only if G is planar and contains any one of the graphs $K_{1,5}$, H_3 or H_4 in Fig.9 as a subgraph.

(3) $\zeta_4(G) = 2$, if and only if $K_{1,4}L(G)$ is planar and G contains any one of the graphs in Fig.10 as a subgraph.

(4) $\zeta_4(G) = \infty$, otherwise. □

Finally, we have the following.

Theorem 3.5. *For $n \geq 5$ and for any graph G, $\zeta_n(G) = \{0, 1, \infty\}$. Also,*

(1) $\zeta_n(G) = 0$, if and only if G is non-planar.

(2) $\zeta_n(G) = \infty$, if and only if G is planar and $\Delta(G) \leq 4$.

(3) $\zeta_n(G) = 1$, otherwise.

Proof. $K_{1,n}L(G), n \geq 5$ will have an edge if and only if $\Delta(G) \geq 5$ and in that case the edges incident on the vertex with maximum degree will induce a K_5 in $K_{1,n}L(G)$ which makes it non-planar. Hence, $\zeta_n(G) = 1$. If G is nonplanar, then $\zeta_n(G) = 0$. If G is planar and $\Delta(G) \leq 4$, then $K_{1,n}L(G), n \geq 5$ is an edgeless graph and hence $\zeta_n(G) = \infty$. □

References

[1] G. Chartrand, H. Gavlas, M. Schultz, H-distance in graphs, *Congr. Numer.*, **101** (1994) 3-15.

[2] M. Ghebleh, M. Khatirinejad, Planarity in iterated line graphs, *Discrete Math.*, **308** (2008) 144-147.

[3] E. B. Jarrett, On iterated triangular line graphs, in: Y.Alavi, A. Schwenk(Eds), *Proc. 7th Internat. Conf. in Graph Theory, Combinatorics, Algorithms, and Applications*, Wiley, New York, (1994).

[4] V. B. Le, Gallai and anti-Gallai graphs, *Discrete Math.*, **159** (1996) 179-189.

[5] Seema Varghese, A.Vijayakumar, *On H-line graphs*, (Submitted).

[6] D.B. West, *Introduction to Graph Theory*, PHI (2003).