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Characterization of $b\gamma$-perfect graphs*

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Abstract

A b-coloring is a proper coloring of the vertices of a graph such that each color class has a vertex that is adjacent to a vertex of every other color, and the b-chromatic number $b(G)$ of a graph G is the largest k such that G admits a b-coloring with k colors. A Grundy coloring is a proper coloring with integers $1, 2, \ldots$ such that every vertex has a neighbor of each color smaller than its own color, and the Grundy number $\gamma(G)$ of a graph G is the largest k such that G admits a Grundy coloring with k colors. An a-coloring (or complete coloring) is a proper coloring of the vertices of a graph such that the union of any two color classes is not an independent set, and the a-chromatic number $\psi(G)$ of a graph G is the largest k such that G admits an a-coloring with k colors. A graph is $b\gamma$-perfect if $b(H) = \gamma(H)$ holds for every induced subgraph of G. We study the relationship between b and γ and characterize $b\gamma$-perfect graphs as a special subclass of P_4-free graphs. We also show how to compute b in polynomial time for every P_4-free graph. We also characterize $b\psi$-perfect graphs.

Keywords: b-coloring, Grundy coloring, P_4-free graphs.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex-set V and edge-set E. A coloring of the vertices of G is a mapping $c : V \to \{1, 2, \ldots\}$. For every vertex v in V the integer $c(v)$ is called the color of v. A coloring is *proper* if any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of graph G is the smallest integer k such that G admits a proper coloring using k colors.

A *b-coloring* is a proper coloring such that every non-empty color class contains a vertex that has a neighbor of each color different from its own color. We call any such vertex a *b-vertex*. The concept of b-coloring was introduced in [5, 8]. The b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. Clearly every coloring of a graph G with $\chi(G)$ colors is a b-coloring, and so every graph G satisfies $\chi(G) \leq b(G)$. Deciding whether a graph G admits a b-coloring with a given number of colors is an NP-complete problem [5, 8], even when it is restricted to the class of bipartite graphs [7]. These NP-completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs [3, 6].

A *Grundy coloring* of a graph G is a coloring with k colors such that every vertex v of color i ($1 \leq i \leq k$) has a neighbor of every color j with $j < i$. The Grundy number of a graph G, denoted by $\gamma(G)$, is the largest integer k such that G admits a Grundy coloring with k colors. This notion was introduced in [2].

An *a-coloring* of a graph G is a proper coloring such that for every pair of distinct colors i and j, there exists two adjacent vertices of color i and j. The *a-chromatic number* of G, denoted by $\psi(G)$, is the largest integer k such that G admits an a-coloring with k colors.

For any two parameters α and β in $\{\chi, b, \gamma, \psi\}$, a graph G is called $\alpha\beta$-perfect whenever $\alpha(H) = \beta(H)$ holds for every induced subgraph H of G. The notion of $b\chi$-perfect graphs was introduced by Hoàng and Kouider in [3] where they described some classes of $b\chi$-perfect graphs. The problem of characterizing $b\chi$-perfect graphs by a list of forbidden subgraphs was solved in [4]. Here we will characterize $b\gamma$-perfect graphs and $b\psi$-perfect graphs.

We finish this section with some definitions and notation. Consider a graph $G = (V, E)$. For any $A \subseteq V$, let $G[A]$ be the subgraph of G induced by A. For any vertex v of G, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$. Let $\omega(G)$ denote the size of a maximum clique of G. If G and H are two vertex-disjoint graphs, the *union* of G and H is the graph $G + H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted by pG. The *join* of graphs G and H is the graph, denoted by $G \vee H$, obtained from $G + H$ by adding all edges between G and H. Given a collection $\mathcal{H}$ of graphs, a graph G is called *$\mathcal{H}$-free* if G does not have an induced subgraph that is isomorphic to any member of $\mathcal{H}$. In case $\mathcal{H}$ has only one member H we say that G is H-free. We let P_k denote the path with k vertices, and K_k denote the complete graph with k vertices. For further terminology on graphs we refer to the book by Berge [1].

2. Lemmas

We will use several results due to Hoàng and Kouider [3] and Kouider and Mahéo [6].

Lemma 2.1. [6] *For integers $n, p \geq 1$, the complete bipartite graph $K_{n,p}$ satisfies $b(K_{n,p}) = 2$.*

Lemma 2.2. [3] *If G_1 and G_2 are any two vertex-disjoint graphs, then $b(G_1 \vee G_2) = b(G_1) + b(G_2)$.*

Lemma 2.3. [6] *If G is the union of graphs $G_1, G_2, \ldots, G_p$, then $b(G) \geq \max\{b(C_i), 1 \leq i \leq p\}$.*

One difficulty frequently encountered in the study of the b-chromatic number is that the inequality shown in the preceding lemma is not always an equality. For example, we have $b(P_3) = b(P_4) = 2$ and $b(P_3 + P_4) = 3$. Some of the lemmas below help deal with this difficulty in special cases.

Lemma 2.4. [3] *Let G be a graph and H a complete graph disjoint from G. Then $b(G + H) = \max\{b(G), b(H)\}$.*

Lemma 2.5. *Let G_1 and G_2 be two vertex-disjoint complete bipartite graphs. Then $b(G_1 + G_2) = 2$.*

Proof. By Lemma 2.1, we have $b(G_i) = 2$ for $i = 1, 2$. Put $G = G_1 + G_2$. By Lemma 2.3, we have $b(G) \geq \max\{b(G_i), 1 \leq i \leq 2\} = 2$. Suppose that $b(G) \geq 3$. Then one of G_1, G_2, say G_1, contains two b-vertices u, v of distinct colors, say colors 1 and 2. Let $G_1 = (X_1, Y_1, E_1)$. We may assume that $u \in X_1$. Then color 1 does not appear in Y_1, so $v \in Y_1$ (else v would have no neighbor of color 1). Since G_1 is a complete bipartite graph, one of X_1, Y_1 contains no vertex of color 3; but then one of u, v has no neighbor of color 3, a contradiction. □

Remark that the equality in the preceding lemma no longer holds if "two" is replaced by "three". For example, it is easy to see that $b(P_3) = b(2P_3) = 2$ and $b(3P_3) = 3$.

For the Grundy number the situation is more straightforward, as shown in the next two lemmas.

Lemma 2.6. *If G_1 and G_2 are any two vertex-disjoint graphs, then $\gamma(G_1 \vee G_2) = \gamma(G_1) + \gamma(G_2)$.*

Proof. Clearly, in every coloring of $G_1 \vee G_2$, no color can appear in both G_1 and G_2. So $\gamma(G_1 \vee G_2) \geq \gamma(G_1) + \gamma(G_2)$. Suppose that $\gamma(G_1 \vee G_2) > \gamma(G_1) + \gamma(G_2)$, and consider a Grundy coloring g of $G_1 \vee G_2$ with $\gamma(G_1 \vee G_2)$ colors. Then, for some $i \in \{1, 2\}$, there are strictly more than $\gamma(G_i)$ colors in G_i, say for $i = 1$. Let $c_1, \ldots, c_h$ be these colors, with $h > \gamma(G_1)$ and $c_1 < \cdots < c_h$. We define a coloring g' of G_1 as follows. For every vertex v of G_1, if $g(v) = c_j$ $(1 \leq j \leq h)$, then set $g'(v) = j$. It is a routine matter to check that g' is a Grundy coloring of G_1 with h colors, which contradicts the definition of $\gamma(G_1)$ because $h > \gamma(G_1)$. □

Lemma 2.7. *If G_1 and G_2 are any two vertex-disjoint graphs, then $\gamma(G_1 + G_2) = \max\{\gamma(G_1), \gamma(G_2)\}$.*

Proof. Put $G = G_1 + G_2$. Clearly, $\gamma(G) \geq \max\{\gamma(G_1), \gamma(G_2)\}$. Suppose that $\gamma(G) > \max\{\gamma(G_1), \gamma(G_2)\}$ and consider a Grundy coloring g of G with $\gamma(G)$ colors. Let v be a vertex with color $g(v) = \gamma(G)$. We may assume that v is in G_1. Let g_1 be the restriction of g to G_1. Then, it is a routine matter to check that g_1 is a Grundy coloring of G_1 with more than $\gamma(G_1)$ colors, a contradiction. □

3. P_4-free graphs

P_4-free graphs will play a major role in our study. We recall a theorem of Seinsche [9] which explains the structure of these graphs.

Theorem 3.1. [9] *A graph G is P_4-free if and only if, for every $A \subseteq V$ with $|A| \geq 2$, either $G[A]$ or its complementary graph $\overline{G}[A]$ is not connected.*

Theorem 3.2. *If G is a P_4-free graph, then $\gamma(G) \leq b(G)$.*

Proof. We prove this theorem by induction on the number n of vertices of G. The theorem holds trivially for $n = 1$. Now suppose that it holds for every k with $k \leq n - 1$. Since G is P_4-free, then, by Theorem 3.1, we can distinguish between two cases:

Case 1. $\overline{G}$ is not connected. Then G is the join of two graphs G_1 and G_2. By Lemmas 2.2 and 2.6, we have $b(G) = b(G_1 \vee G_2) = b(G_1) + b(G_2) \geq \gamma(G_1) + \gamma(G_2) = \gamma(G_1 \vee G_2) = \gamma(G)$.

Case 2. G is not connected. Then G is the union of $p \geq 2$ graphs $G_1, \ldots, G_p$. By Lemmas 2.3 and 2.7, we have $b(G) = b(G_1 + G_2 + \cdots + G_p) \geq \max\{b(G_i) \mid 1 \leq i \leq p\} \geq \max\{\gamma(G_i) \mid 1 \leq i \leq p\} = \gamma(G)$. □

Remark that the converse of the preceding theorem does not hold. For example, Figure 1 shows two pictures of a graph G with $\gamma(G) = 4$, $b(G) = 5$, and G contains a P_4.

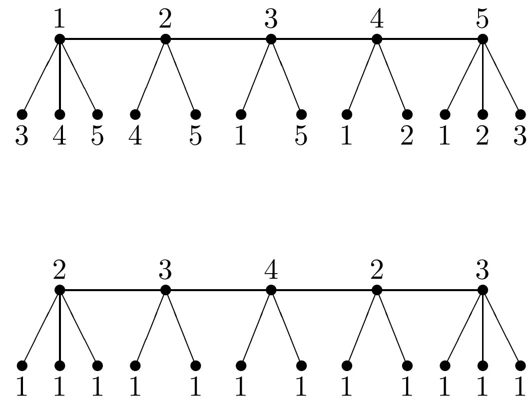

Figure 1: The first picture shows a b-coloring of G with 5 colors. The second picture shows a Grundy coloring of G with 4 colors.

3.1. Computing the b-chromatic number

We show here a polynomial time algorithm to compute the b-chromatic number of a P_4-free graph G. For this purpose let us introduce some additional definitions. In a given coloring, we say that a color class is a *b-class* if it contains a b-vertex. For every integer q with $\chi(G) \le q \le |V(G)|$, let $h(G, q)$ be the largest number of b-classes in all colorings with q non-empty colors. With this notation, we see that

$$b(G) = \max_{\chi(G) \le q \le |V(G)|} \{q \mid h(G, q) = q\}.$$

So, in order to compute $b(G)$, it suffices to compute $h(G, q)$ for every integer $q \in \{\chi(G), \ldots, |V(G)|\}$. If G has only one vertex, then $q = 1$ and $h(G, 1) = 1$. Now assume that G has at least two vertices. By Theorem 3.1, we distinguish between two cases.

Case 1. G is not connected. So G is the union of two graphs G_1, G_2. We claim that

$$h(G, q) = \min\{q, h(G_1, q) + h(G_2, q)\}.$$

Clearly, $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$. Let us consider a coloring c of G with exactly q colors and $r = h(G, q)$ b-classes. For $j = 1, 2$, let r_j be the number of b-classes of c that are included in G_j; so $r \le r_1 + r_2$. Suppose that $r_j > 0$. Then G_j contains a b-vertex u. Since u has neighbors of all colors other than $c(u)$, and since all neighbors of u are in G_j, the graph G_j has vertices of all q colors, and all those b-vertices of c that are in G_j are b-vertices of the restriction of c to G_j, so $h(G_j, q) \ge r_j$. We deduce that $h(G, q) = r \le r_1 + r_2 \le h(G_1, q) + h(G_2, q)$. Conversely, for $j = 1, 2$, let c_j be a coloring of G_j with q colors and $r_j = h(G_j, q)$ b-classes. Without loss of generality we may rename the colors so that the b-classes of c_1 are colors $1, \ldots, r_1$ and the b-classes of c_2 are colors $q - r_2 + 1, \ldots, q$. Consequently, by combining c_1 and c_2 we obtain a coloring of G with q colors and $\min\{q, r_1 + r_2\}$ b-classes. So $h(G, q) \geq \min\{q, h(G_1, q) + h(G_2, q)\}$. The two inequalities imply the equality claimed above.

Case 2. $\overline{G}$ is not connected. So G is the join of two graphs G_1, G_2. We claim that

$$h(G, q) = \max\{h(G_1, q_1) + h(G_2, q_2) \mid q_1 \geq \chi(G_1), q_2 \geq \chi(G_2), q_1 + q_2 = q\}.$$

It is clear that $\chi(G) = \chi(G_1) + \chi(G_2)$. Let us consider a coloring c of G with exactly q non-empty colors and $r = h(G, q)$ b-classes. For $j = 1, 2$, let q_j and r_j be respectively the number of colors and of b-classes of c that are in G_j. Since a color cannot appear in both G_1 and G_2, we have $q = q_1 + q_2$ and $r = r_1 + r_2$. Every b-vertex of c that lies in G_j is a b-vertex of the restriction of c to G_j, so $r_j \leq h(G_j, q_j)$ $(j = 1, 2)$. Thus $h(G, q) = r_1 + r_2 \leq h(G_1, q_1) + h(G_2, q_2)$. Conversely, let q_1, q_2 be two integers such that $q_1 \geq \chi(G_1)$, $q_2 \geq \chi(G_2)$ and $q_1 + q_2 = q$. For $j = 1, 2$, put $r_j = h(G_j, q_j)$ and let c_j be a coloring of G_j with q_j colors and r_j b-classes. By assigning disjoint sets of colors to c_1 and c_2, their combination is a coloring c of G with q colors. Since every vertex of G_1 is adjacent to every vertex of G_2, it follows that, for each j in $\{1, 2\}$, every vertex of G_j that is a b-vertex of c_j is also a b-vertex of c. So c has $r_1 + r_2$ b-classes. Since this holds for all possible choices of q_1, q_2, we obtain $h(G, q) \geq \max\{h(G_1, q_1) + h(G_2, q_2) \mid q_1 \geq \chi(G_1), q_2 \geq \chi(G_2), q_1 + q_2 = q\}$. The two inequalities imply the equality claimed above.

We observe that all operations in this proof can be performed in polynomial time. In Case 1, the computation of $h(G, q)$ is immediate from $h(G_1, q)$ and $h(G_2, q)$. In Case 2, at most $|V(G)|$ couples (q_1, q_2) must be examined. Moreover, each of Case 1 and 2 reduces the computation of $h(G, q)$ to the computation on two disjoint subgraphs. Thus we have at most $|V(G)|$ occurrences of Case 1 and 2. In total, the computation time is $O(|V(G)|^2)$ for every value of q, and so $O(|V(G)|^3)$ for all possible values and in particular to deduce the exact value of $b(G)$ and to construct a b-coloring of G with $b(G)$ colors.

4. Characterisation of $b\gamma$-perfect graphs

In the preceding section we saw that $\gamma(G) \leq b(G)$ holds for P_4-free graphs. However, in general the b-chromatic number and the Grundy number are incomparable. For example, if D is the *diamond* (the simple graph with four vertices and five edges), and $2D$ is the union of two diamonds, then we have $\gamma(2D) = 3$ and $b(2D) = 4$; on the other hand we have $\gamma(P_4) = 3$, and $b(P_4) = 2$. So it is interesting to compare these two numbers for particular classes of graphs and to characterize the class of $b\gamma$-perfect graphs.

Remark that if a graph G admits a b-coloring with k colors, then G has at least k vertices of degree at least $k - 1$. Irving and Manlove [5, 8] defined the number $m(G)$ of a graph G as the largest integer h such that G has at least h vertices of degree at least $h - 1$. We will call this the m-degree of G. Thus every graph satisfies $\omega(G) \leq b(G) \leq m(G)$.

Lemma 4.1. *Let G be a $\{P_4, 2D, 3P_3\}$-free graph that is the union of $p \geq 2$ graphs $G_1, \ldots, G_p$. Then $b(G) = \max\{b(G_1), \ldots, b(G_p)\}$.*

Proof. We prove the lemma by induction on p. First suppose $p = 2$. Since G is $2D$-free, we may suppose that one of G_1, G_2, say G_1, is diamond-free. If G_1 is a clique, then Lemma 2.4 implies the desired equality. Assume now that G_1 is not a clique. So G_1 contains a P_3, and so G_2 is $2P_3$-free. Suppose that $b(G) > \max\{b(G_1), b(G_2)\}$. Put $b(G) = k$. We have $k \geq 3$ (because $b(G_1) \geq 2$). Let c be a b-coloring of G with k colors, and let $u_1, u_2, \ldots, u_k$ be b-vertices of colors $1, 2, \ldots, k$ respectively. Graph G_2 cannot contain b-vertices of all colors, for otherwise c would be a b-coloring of G_2, contradicting $b(G_2) < k$. So we may suppose that G_2 contains no b-vertex of color 1, and so u_1 is in G_1. We claim that:

Vertices $u_2, \ldots, u_k$ are in G_2.

To prove this, let us examine the structure of G_1. Since G_1 is P_4-free, it is the join of two graphs $G[A]$ and $G[B]$. Subgraphs $G[A]$ and $G[B]$ are P_3-free, for otherwise G_1 contains a diamond. Since G_1 is not a clique, we may assume up to symmetry that A has two non-adjacent vertices. Thus $G[A]$ is the union of at least two cliques. Then B is independent, for otherwise G_1 contains a diamond. Suppose that $|B| \geq 2$. Then, by a symmetric argument, A is independent, and so G_1 is a complete bipartite graph. We may assume that $u_1 \in A$. Then B contains vertices of each color $2, \ldots, k$, so all vertices of A have color 1, and consequently G_1 contains no b-vertex of color $2, \ldots, k$. Thus $u_2, \ldots, u_k$ are in G_2. Now suppose that $|B| = 1$. Then it is easy to check that the m-degree of G_1 is equal to $\omega(G_1)$. So, by the remark before the lemma, we have $\omega(G_1) = b(G_1) \leq k - 1$. It follows that every vertex of A has degree at most $\omega(G_1) - 1 \leq b(G) - 2$. Consequently, no vertex of A can be a b-vertex (and $B = \{u_1\}$). Thus $u_2, \ldots, u_k$ are in G_2. So the claim is proved.

Note that G_2 contains vertices of color 1. Let v be a vertex of G_2 of color 1 with the largest possible number of neighbors in the set $U = \{u_2, \ldots, u_k\}$. Vertex v cannot be adjacent to all of U, for otherwise v would be a b-vertex of color 1 in G_2; so v is not adjacent to some vertex u_j in U. Since u_j is a b-vertex, it has a neighbor w of color 1. The choice of v implies that there exists a vertex u_i in U that is adjacent to v and not to w. Vertices u_i and u_j are not adjacent, for otherwise v-u_i-u_j-w would be a P_4. Vertex v cannot be adjacent to all neighbors of u_i of color $\neq 1$, for otherwise v would be a b-vertex. So there exists a vertex x adjacent to u_i and not to v. Similarly, there exists a vertex y adjacent to u_j and not to w. Vertex x is not adjacent to u_j, for otherwise u_j-x-u_i-v is a P_4. Likewise y is not adjacent to u_i. So $x \neq y$. Then xy, xw and yv are not edges of G, for otherwise u_i-x-y-u_j, w-x-u_i-v or v-y-u_j-w are P_4's. Now x-u_i-v and w-u_j-y form a $2P_3$ in G_2, a contradiction.

Now suppose that $p \geq 3$. Since G is $3P_3$-free, at least one of G_1, G_2, G_3, say G_1, is P_3-free; and so G_1 is a clique. Then, by Lemma 2.4, we have $b(G) = \max\{b(G_1), b(G - G_1)\}$, and by the induction hypothesis we have $b(G - G_1) = \max\{b(G_2), \ldots, b(G_p)\}$. So $b(G) = \max\{b(G_1), \ldots, b(G_p)\}$. □

Theorem 4.2. *A graph is $b\gamma$-perfect if and only if it is $\{P_4, 3P_3, 2D\}$-free.*

Proof. It is easy to check that $b(P_4) = 2$, $\gamma(P_4) = 3$, $b(3P_3) = 3$, $\gamma(3P_3) = 2$, $b(2D) = 4$ and $\gamma(2D) = 3$. So a $b\gamma$-perfect graph cannot contain any of these three graphs. Conversely, let G be any $\{P_4, 3P_3, 2D\}$-free graph. We will prove that G is $b\gamma$-perfect by induction on the number n of vertices of G. If $n = 1$ the fact is obvious, so let us suppose that $n \geq 2$. Since G is P_4-free, by Theorem 3.1, we distinguish between two cases.

Case 1. $\overline{G}$ is not connected. Then G is the join of two graphs G_1 and G_2. By Lemmas 2.2 and 2.6, we have $b(G) = b(G_1 \vee G_2) = b(G_1) + b(G_2)$ and $\gamma(G) = \gamma(G_1 \vee G_2) = \gamma(G_1) + \gamma(G_2)$. By the induction hypothesis, G_1 and G_2 are $b\gamma$-perfect. Thus we obtain $b(G) = \gamma(G)$.

Case 2. G is not connected. Then G is the union of p graphs $G_1, \ldots, G_p$ with $p \geq 2$. By Lemma 2.7, we have $\gamma(G) = \max\{\gamma(G_1), \ldots, \gamma(G_p)\}$. By Lemma 4.1 we have $b(G) = \max\{b(G_1), \ldots, b(G_p)\}$. By the induction hypothesis, $G_1, \ldots, G_p$ are $b\gamma$-perfect. Thus we obtain $b(G) = \gamma(G)$. □

5. $b\psi$-perfect graphs

Christen and Selkow [2] characterized $\psi\chi$-perfect graphs.

Theorem 5.1. [2] *A graph is $\psi\chi$-perfect if and only if it is $\{P_4, 3P_2, P_3 + P_2\}$-free.*

Using Theorem 5.1, we can deduce the following corollary.

Corollary 5.2. *For a graph G, the following conditions are equivalent:*

(i) G is $b\psi$-perfect.

(ii) G is $\{P_4, 3P_2, P_3 + P_2\}$-free.

(iii) G is $\psi\chi$-perfect.

Proof. (i)⇒(ii): Clearly, $b(P_4) = 2$, $\psi(P_4) = 3$, $b(3P_2) = 2$, $\psi(3P_2) = 3$ and $b(P_3 + P_2) = 2$, $\psi(P_3 + P_2) = 3$. So a $b\psi$-perfect graph cannot contain any of these three graphs.

(ii)⇒(iii) follows from Theorem 5.1.

(iii)⇒(i) is trivial because every graph G satisfies $\chi(G) \leq b(G) \leq \psi(G)$. □

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