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On total edge irregularity strength of categorical product of cycle and path

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Abstract

We investigate a modification of well known irregularity strength of graph, namely the total edge irregularity strength. An edge irregular total k-labeling $\varphi : V \cup E \to \{1, 2, \dots, k\}$ of a graph G is a labeling of vertices and edges of G in such a way that for any two different edges uv and $u'v'$ their weights $\varphi(u) + \varphi(uv) + \varphi(v)$ and $\varphi(u') + \varphi(u'v') + \varphi(v')$ are distinct. The total edge irregularity strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

The main purpose of this paper is to solve the open problem posed by Ahmad and Bača.

Keywords: Irregularity strength, total edge irregularity strength, irregular assignment, edge irregular total labeling, the categorial product of cycle and path.

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1. Introduction

We consider finite undirected graphs $G = (V, E)$ without loops and multiple edges with vertex-set $V(G)$ and edge-set $E(G)$, where $|V(G)| = p$ and $|E(G)| = q$. The degree of a vertex x is the number of edges that have x as an endpoint, and the set of neighbors of x is denoted by $N(x)$. By a *labeling* we mean any mapping that carries a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V \cup E$ then we call the labeling *total labeling*. For an edge k-labeling $\sigma : E(G) \to \{1, 2, \dots, k\}$ the associated vertex-weight of a vertex $x \in V(G)$ is

$$w_\sigma(x) = \sum_{y \in N(x)} \sigma(xy)$$

and for a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ the associated edge-weight is

$$wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y).$$

Chartrand et al. in [8] introduced edge k-labeling of a graph G such that $w(x) \neq w(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

This parameter has attracted much attention in [3], [4], [6], [10], [14]. Motivated by these papers Bača et al. in [5] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labeling.

For a graph $G = (V, E)$ they define a labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ to be an *edge irregular total k-labeling* if for every two different edges xy and $x'y'$ of G the *edge-weights* $wt_\varphi(xy) \neq wt_\varphi(x'y')$. The *total edge irregularity strength*, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

Let φ be an edge irregular total k-labeling of $G = (V, E)$. Since $3 \leq wt_\varphi(xy) \leq 3k$ for every edge $xy \in E(G)$, we have $|E(G)| \leq 3k - 2$, which implies $tes(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil$.

If $x \in V(G)$ is a fixed vertex of maximum degree $\Delta(G)$, then there is a range of $2k - 1$ possible weights $\varphi(x) + 2 \leq wt_\varphi(xy) \leq \varphi(x) + 2k$ for the $\Delta(G)$ edges $xy \in E(G)$ incident with x which implies $tes(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. So, we have that

$$tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}. \tag{1}$$

The authors of [5] determined the exact value of the total edge irregularity strength for certain families of graphs, namely paths, cycles, stars, wheels and friendship graphs. They posed the problem to determine the total edge irregularity strength of trees. Recently Ivančo and Jendroľ [11] proved that for any tree T the total edge irregularity strength is equal to its lower bound, i.e.

$$tes(T) = \max \left\{ \left\lceil \frac{|E(T)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(T) + 1}{2} \right\rceil \right\}.$$

Moreover, they posed the following conjecture.

Conjecture 1.1. [11] *Let $G = (V, E)$ be an arbitrary graph different from K_5 and maximum degree $\Delta(G)$. Then*

$$tes(G) = \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}.$$

Conjecture 1 has been verified for trees in [11], for complete graphs and complete bipartite graphs in [12] and [13], for the Cartesian product of two paths in [15], for corona product of a path with certain graphs in [16], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [7], for the categorical product of two paths in [2] and for the categorical product of a cycle and a path for $m,\ n$ even in [1]. Motivated by the papers [9] and [15] Ahmad and Bača in [1] investigated the total edge irregularity strength of the categorial product of two graphs.

The categorial product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices $(u, \acute{u})$ and $(v, \acute{v})$ are adjacent if and only if u, v are adjacent in G and $\acute{u}, \acute{v}$ are adjacent in H (see e.g [17] or [18]).

For integers a and b let $[a, b]$ be an interval of integers c, $a \leq c \leq b$. If we consider graph G as a cycle C_n with $V(C_n) = \{a_i : i \in [1, n]\}$, $E(C_n) = \{a_i a_{i+1} : i \in [1, n-1]\} \cup \{a_n a_1\}$ and graph H as a path P_m with $V(P_m) = \{b_j : j \in [1, m]\}$, $E(P_m) = \{b_j b_{j+1} : j \in [1, m-1]\}$, then $V(C_n \times P_m) = \{(a_i, b_j) : i \in [1, n],\ j \in [1, m]\}$ is the vertex set of the graph $C_n \times P_m$ and $E(C_n \times P_m) = \{(a_i, b_j)(a_{i+1}, b_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \cup \{(a_{i+1}, b_j)(a_i, b_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \cup \{(a_1, b_j)(a_n, b_{j+1}) : j \in [1, m-1]\} \cup \{(a_n, b_j)(a_1, b_{j+1}) : j \in [1, m-1]\}$ is the edge set of $C_n \times P_m$. So $C_n \times P_m$ is a graph of order nm and size $2n(m-1)$.

In [1], Ahmad and Bača posed the following open problem.

Open Problem 1.2. [1] *For the categorical product $C_n \times P_m$, $m \geq 2$, $n \geq 3$, determine the total edge irregularity strength if at least one of m, n is odd.*

In this paper, we solve this open problem. As the maximum degree $\triangle(C_n \times P_m) = 4$, then (1) implies that $tes(C_n \times P_m) \geq \left\lceil \frac{2n(m-1)+2}{3} \right\rceil$. To show that $\left\lceil \frac{2n(m-1)+2}{3} \right\rceil$ is an upper bound for the $tes(C_n \times P_m)$, we describe an edge irregular total $\left\lceil \frac{2n(m-1)+2}{3} \right\rceil$-labeling for $C_n \times P_m$.

2. Total edge irregularity strength for small cases

In this section we discuss the total edge irregularity strength for $C_4 \times P_m,\ m \geq 3$ odd and $C_5 \times P_m,\ m \geq 2$.

Lemma 2.1. *Let $m \geq 3$ be an odd integer. Then $tes(C_4 \times P_m) = \left\lceil \frac{8m-6}{3} \right\rceil$.*

Proof. Let $k = \left\lceil \frac{8m-6}{3} \right\rceil$. From (1) it follows that $tes(C_4 \times P_m) \geq k$. To prove equality, we describe the total labeling ξ_1, for $1 \leq j \leq m$, as follows .

$$\xi_1((a_i, b_j)) = \begin{cases} j, & \text{if } i = 1 \\ 1, & \text{if } i = 2 \\ k - m + j, & \text{if } i = 3 \\ k, & \text{if } i = 4 \end{cases}$$

For $1 \leq j \leq m-1$, we define

$$\xi_1((a_i, b_j)(a_{i+1}, b_{j+1})) = \begin{cases} 1, & \text{if } i = 1 \\ 3m - k - 2, & \text{if } i = 2 \\ 7m - 2k - 4, & \text{if } i = 3 \end{cases}$$

$$\xi_1((a_{i+1}, b_j)(a_i, b_{j+1})) = \begin{cases} m - 1, & \text{if } i = 1 \\ 4m - k - 2, & \text{if } i = 2 \\ 8m - 2k - 6, & \text{if } i = 3 \end{cases}$$

$\xi_1((a_1, b_j)(a_4, b_{j+1})) = 4m - k - 2$, $\xi_1((a_4, b_j)(a_1, b_{j+1})) = 5m - k - 4$.

We can see that all vertex and edge labels are at most k and moreover under the total labeling ξ_1 the edges:

(i) from the set $\{(a_1, b_j)(a_2, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[3, m+1]$,

(ii) from the set $\{(a_2, b_j)(a_1, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[m+2, 2m]$,

(iii) from the set $\{(a_2, b_j)(a_3, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[2m+1, 3m-1]$,

(iv) from the set $\{(a_3, b_j)(a_2, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[3m, 4m-2]$,

(v) from the set $\{(a_1, b_j)(a_4, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[4m-1, 5m-3]$,

(vi) from the set $\{(a_4, b_j)(a_1, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[5m-2, 6m-4]$,

(vii) from the set $\{(a_3, b_j)(a_4, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[6m-3, 7m-5]$,

(viii) from the set $\{(a_4, b_j)(a_3, b_{j+1}) : j \in [1, m-1]\}$ admit the edge-weights from the interval $[7m-4, 8m-6]$.

It is easy to verify that under the total labeling ξ_1 the edge-weights are distinct for all pairs of distinct edges. This completes the proof. □

Lemma 2.2. *Let $m \geq 2$. Then $tes(C_5 \times P_m) = \lceil \frac{10m-8}{3} \rceil$.*

Proof. Let $k = \lceil \frac{10m-8}{3} \rceil$. With respect to (1) it is sufficient to find an optimal edge irregular total k-labeling. For $1 \leq j \leq m$ define the vertex labeling ξ_2 as follows:

$$\xi_2((a_i, b_j)) = \begin{cases} j, & \text{if } i = 1 \\ (i-2)(1 + \lceil \frac{m}{2} \rceil) + \lceil \frac{j}{2} \rceil, & \text{if } i = 2, 3 \\ k - \lceil \frac{m}{2} \rceil + \lceil \frac{j}{2} \rceil, & \text{if } i = 4 \\ k, & \text{if } i = 5 \end{cases}$$

For $1 \leq j \leq m-1$, we construct the edge labeling in the following way:

$$\xi_2((a_i, b_j)(a_{i+1}, b_{j+1})) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{if } i = 1 \\ m - 2 + \lceil \frac{m-1}{2} \rceil, & \text{if } i = 2 \\ 6(m-1) - k, & \text{if } i = 3 \\ 8(m-1) - 2k + 1 + \lceil \frac{m}{2} \rceil + \lceil \frac{j+1}{2} \rceil, & \text{if } i = 4 \end{cases}$$

$$\xi_2((a_{i+1}, b_j)(a_i, b_{j+1})) = \begin{cases} 1 + \lceil \frac{j-1}{2} \rceil, & \text{if } i = 1 \\ 2m - 3 + \lceil \frac{m-1}{2} \rceil, & \text{if } i = 2 \\ 7(m-1) - k, & \text{if } i = 3 \\ 9(m-1) - 2k + 1 + \lceil \frac{m}{2} \rceil + \lceil \frac{j}{2} \rceil, & \text{if } i = 4 \end{cases}$$

$\xi_2((a_1, b_j)(a_5, b_{j+1})) = 4m - k - 2$, $\xi_2((a_5, b_j)(a_1, b_{j+1})) = 5m - k - 4$.

It is easy to see that under the labeling ξ_2 the vertices and edges admit labels from the set $\{1, 2, \ldots, k\}$. Moreover the edge-weights, for $1 \leq j \leq m-1$, are as follows:

$wt((a_1, b_j)(a_2, b_{j+1})) = 2j + 1,\ wt((a_2, b_j)(a_1, b_{j+1})) = 2j + 2,$

$wt((a_2, b_j)(a_3, b_{j+1})) = 2m + j,\ wt((a_3, b_j)(a_2, b_{j+1})) = 3m - 1 + j,$

$wt((a_1, b_j)(a_5, b_{j+1})) = 4m - 2 + j$, $wt((a_5, b_j)(a_1, b_{j+1})) = 5m - 3 + j$,

$wt((a_3, b_j)(a_4, b_{j+1})) = 6m - 4 + j$, $wt((a_4, b_j)(a_3, b_{j+1}) = 7m - 5 + j$,

$wt((a_4, b_j)(a_5, b_{j+1})) = 8m - 6 + j$, $wt((a_5, b_j)(a_4, b_{j+1})) = 9m - 7 + j$.

It is easy to verify that under the labeling ξ_2 the edge-weights are distinct for all pairs of distinct edges and constitute the arithmetic progression of consecutive integers from 3 to $10m - 8$. This completes the proof. □

3. Main Result

In this section we describe an optimal edge irregular total $\left\lceil\frac{2n(m-1)+2}{3}\right\rceil$-labeling for categorical product of cycle and path, for $n \geq 6$ and $m \geq 2$.

Theorem 3.1. *Let $m \geq 2, n \geq 4$ be two integers, not even simultaneously and $C_n \times P_m$ be the categorical product of the cycle C_n and the path P_m. Then*

$$tes(C_n \times P_m) = \left\lceil\frac{2n(m-1)+2}{3}\right\rceil.$$

Proof. The cases when $n = 4,\ m \geq 3$, and $n = 5,\ m \geq 2$, were discussed in the previous section. Now we suppose that $n \geq 6,\ m \geq 2$ are not even simultaneously and that $k = \left\lceil\frac{2n(m-1)+2}{3}\right\rceil$.
We split the edge set of $C_n \times P_m$ in mutually disjoint subsets:
$A_i = \{(a_i, b_j)(a_{i+1}, b_{j+1}) : j \in [1, m-1]\}$,
$A'_i = \{(a_{i+1}, b_j)(a_i, b_{j+1}) : j \in [1, m-1]\}$ for $i \in [1, n-1]$,
$B_j = \{(a_1, b_j)(a_n, b_{j+1})\}$, $B'_j = \{(a_n, b_j)(a_1, b_{j+1})\}$ for $j \in [1, m-1]$.

For $j \in [1, m]$, define a vertex labeling ψ as follows:

$$\psi((a_i, b_j)) = \begin{cases} j, & \text{if } i = 1 \\ (i-2)\lceil\frac{m}{2}\rceil + \lceil\frac{j}{2}\rceil, & \text{if } 2 \leq i \leq \lceil\frac{n}{2}\rceil \\ k - (n-i)\lceil\frac{m}{2}\rceil + \lceil\frac{j}{2}\rceil, & \text{if } \lceil\frac{n}{2}\rceil + 1 \leq i \leq n-1 \\ k, & \text{if } i = n \end{cases}$$

It is easy to see that under the vertex labeling the edges:

(i) from the set A_1 admit the weights $\lceil\frac{3j}{2}\rceil$, for j odd, and $\lceil\frac{3j+1}{2}\rceil$, for j even,

(ii) from the set A'_1 admit the weights $1 + \lceil\frac{3j}{2}\rceil$, for all $1 \leq j \leq m-1$,

(iii) from the set A_i and A'_i, for $2 \leq i \leq \lceil\frac{n}{2}\rceil - 1$, admit the weights from the interval $[(2i-3)\lceil\frac{m}{2}\rceil + 2,\ 2(i-1)\lceil\frac{m}{2}\rceil + \lceil\frac{m-1}{2}\rceil]$,

(iv) from the set $A_{\lceil\frac{n}{2}\rceil}$ and $A'_{\lceil\frac{n}{2}\rceil}$ admit the weights from the interval $[k + (\lceil\frac{n}{2}\rceil - \lfloor\frac{n}{2}\rfloor - 1)\lceil\frac{m}{2}\rceil + 2,\ k + (\lceil\frac{n}{2}\rceil - \lfloor\frac{n}{2}\rfloor)\lceil\frac{m}{2}\rceil + \lceil\frac{m-1}{2}\rceil]$,

(v) from the set A_i and A'_i, for $\lceil\frac{n}{2}\rceil + 1 \leq i \leq n-2$, admit the weights from the interval $[2k - (2n-2i-1)\lceil\frac{m}{2}\rceil + 2,\ 2k - 2(n-i-1)\lceil\frac{m}{2}\rceil + \lceil\frac{m-1}{2}\rceil]$,

(vi) from the set A_{n-1} (respectively, A'_{n-1}) admit the weights $2k - \lceil\frac{m}{2}\rceil + \lceil\frac{j}{2}\rceil$ (respectively, $2k - \lceil\frac{m}{2}\rceil + \lceil\frac{j+1}{2}\rceil$), for $j \in [1, m-1]$,

(vii) from the set B_j (respectively, B'_j) admit the weights from the interval $[k+1,\ k+m-1]$ (respectively, $[k+2,\ k+m]$)

For $j \in [1, m-1]$, construct an edge labeling in the following way:

if an edge e_A belongs to A_i then

$$\psi(e_A) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{if } i = 1 \\ m + (2i-3)\lceil \frac{m-3}{2} \rceil, & \text{if } i \in [2, \lceil \frac{n}{2} \rceil - 1] \\ 2(\lfloor \frac{m}{2} \rfloor - 1)\lceil \frac{n}{2} \rceil + (n+1)\lceil \frac{m}{2} \rceil - k + 1, & \text{if } i = \lceil \frac{n}{2} \rceil \\ 2(m-1)\lceil \frac{n}{2} \rceil + (2\lfloor \frac{n}{2} \rfloor - 1)\lceil \frac{m}{2} \rceil & \text{if } i \in [\lceil \frac{n}{2} \rceil + 1, n-2] \\ +(2i - 2\lceil \frac{n}{2} \rceil)\lceil \frac{m-3}{2} \rceil - 2k + 1, & \\ 3 - 2(k+n) + (2\lfloor \frac{n}{2} \rfloor - 1)\lceil \frac{m}{2} \rceil & \text{if } i = n-1 \\ +2m\lceil \frac{n}{2} \rceil + 2(\lfloor \frac{n}{2} \rfloor - 1)\lceil \frac{m-1}{2} \rceil + \lceil \frac{j+1}{2} \rceil, & \end{cases}$$

if an edge e'_A belongs to A'_i then

$$\psi(e'_A) = \begin{cases} \lceil \frac{j+1}{2} \rceil, & \text{if } i = 1 \\ \lceil \frac{3m}{2} \rceil + (2i-2)\lceil \frac{m-3}{2} \rceil, & \text{if } i \in [2, \lceil \frac{n}{2} \rceil - 1] \\ 2(\lfloor \frac{m}{2} \rfloor - 1)\lceil \frac{n}{2} \rceil + (n+2)\lceil \frac{m}{2} \rceil & \text{if } i = \lceil \frac{n}{2} \rceil \\ -k + \lceil \frac{m-1}{2} \rceil, & \\ 2(m-1)\lceil \frac{n}{2} \rceil + 2\lfloor \frac{n}{2} \rfloor \lceil \frac{m}{2} \rceil - 2k & \text{if } i \in [\lceil \frac{n}{2} \rceil + 1, n-2] \\ +(2i - 2\lceil \frac{n}{2} \rceil + 1)\lceil \frac{m-3}{2} \rceil + 1, & \\ 2n\lceil \frac{m}{2} \rceil + (2\lfloor \frac{n}{2} \rfloor - 1)\lceil \frac{m-1}{2} \rceil + 2 & \text{if } i = n-1 \\ +2\lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil - 2(k+n) + \lceil \frac{j}{2} \rceil, & \end{cases}$$

if an edge e_B belongs to B_j and an edge e'_B belongs to B'_j then

$\psi(e_B) = 2m(\lceil \frac{n}{2} \rceil - 1) - 2\lceil \frac{n}{2} \rceil - k + 4$, for all $j \in [1, m-1]$,

$\psi(e'_B) = m(2\lceil \frac{n}{2} \rceil - 1) - 2\lceil \frac{n}{2} \rceil - k + 2$, for all $j \in [1, m-1]$.

Under the total labeling ψ the edges:

(i) from the set A_1 receive the edge-weights $2j+1$,

(ii) from the set A_i, for $2 \le i \le \lceil \frac{n}{2} \rceil - 1$, admit the edge-weights from the interval $[(2m-2)(i-1)+3, (m-1)(2i-1)+2]$,

(iii) from the set $A_{\lceil \frac{n}{2} \rceil}$ admit the edge-weights from the interval $[2\lceil \frac{n}{2} \rceil(m-1)+3, (2\lceil \frac{n}{2} \rceil + 1)(m-1)+2]$,

(iv) from the set A_i, for $\lceil \frac{n}{2} \rceil + 1 \le i \le n-2$, admit the edge-weights from the interval $[2i(m-1)+3, (2i+1)(m-1)+2]$,

(v) from the set A_{n-1} admit the edge-weights from the interval $[(m-1)(2n-2)+3, (2n-1)(m-1)+2]$,

(vi) from the set A'_1 receive the edge-weights $2j+2$,

(vii) from the set A'_i, for $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, admit the edge-weights from the interval $[(2i-1)(m-1)+3, 2i(m-1)+2]$,

(viii) from the set $A'_{\lceil \frac{n}{2} \rceil}$ admit the edge-weights from the interval $[(2\lceil \frac{n}{2} \rceil + 1)(m-1) + 3, (\lceil \frac{n}{2} \rceil + 1)(2m-2)+2]$,

(ix) from the set A'_i, for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n-2$, admit the edge-weights from the interval $[(2i+1)(m-1)+3, (i+1)(2m-2)+2]$,

(x) from the set A'_{n-1} admit the edge-weights from the interval $[(2n-1)(m-1) + 3, 2n(m-1)+2]$,

(xi) from the set B_j admit the edge-weights from the interval $[(\lceil \frac{n}{2} \rceil - 1)(2m-2) + 3, (2\lceil \frac{n}{2} \rceil - 1)(m-1)+2]$,

(xii) from the set B'_j admit the edge-weights from the interval $[(2\lceil \frac{n}{2} \rceil - 1)(m-1) + 3, 2\lceil \frac{n}{2} \rceil(m-1)+2]$.

It is easy to verify that all vertex and edge labels are at most k and the edge-weights are different for all pairs of distinct edges. Thus, the resulting total labeling is the desired edge irregular k-labeling. □

4. Conclusion

In this paper we have determined the exact value of the total edge irregularity strength of categorical product $C_n \times P_m$, for $m \geq 2$, $n \geq 4$ (m, n are not even simultaneously).

We are not able to give an answer as to whether or not there exists an edge irregular total $\left\lceil \frac{6m-4}{3} \right\rceil$-labeling of $C_3 \times P_m$, for all $m \geq 2$. We know such labeling only for several small values of m, but general labeling for all $m \geq 2$ is not known. Therefore, we pose the following open problem.

Open Problem 4.1. *For the categorical product $C_3 \times P_m$, $m \geq 2$, determine the total edge irregularity strength.*

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