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Hedetniemi's Conjecture and Fiber Products of Graphs

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Abstract

We prove that for $n \geq 4$, a fiber product of n-chromatic graphs over n-colourings can have chromatic number strictly less than n. This refutes a conjecture of A. Khelladi, Sur une opération binaire de graphes, Communication to the 32d Week of Science, Damas (Syria, 7-13 November 1992).

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1. Introduction

The *categorical product* of two graphs G and H is the graph $G \times H$ with vertex-set $V(G \times H) = V(G) \times V(H)$, whose edges are the pairs $\{(u, u'), (v, v')\}$ such that $\{u, v\}$ is an edge of G and $\{u', v'\}$ is an edge of H. The chromatic number of a categorical product of graphs is the object of a long standing conjecture.

Conjecture 1.1. [4]

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}. \tag{1}$$

El-Zahar and Sauer [3] proved that equality holds in (1) when $\min\{\chi(G), \chi(H)\} = 4$. More precisely, they proved the following.

Theorem 1.2. [3] *Let G and H be 4-chromatic connected graphs, and G', H' odd cycles contained in G and H respectively. Let A be the subgraph of $G \times H$ induced by*

$$V(A) = \{(u, v) \in V(G \times H) : u \in V(G') \text{ or } v \in V(H')\}.$$

Then $\chi(A) = 4$.

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El-Zahar and Sauer conjectured that a similar result would hold for larger chromatic numbers. Unfortunately, this turned out to be false.

Theorem 1.3. [10] *Let n, m be integers such that $4 \le n < m$. Then there exist m-chromatic connected graphs graphs G, H containing n-chromatic subgraphs G' and H' respectively such that the subgraph of $G \times H$ induced by*

$$\{(u,v) \in V(G \times H) : u \in V(G') \text{ or } v \in V(H')\}$$

has chromatic number n.

The fiber product provides an alternative hypothesis for the structure of n-chromatic subgraphs of n-chromatic graphs. Let G, H be two n-colourable graphs, and $c_G : G \to K_n$, $c_H : H \to K_n$ proper n-colourings of G and H respectively. The *fiber product* of G and H with respect to c_G and c_H is the subgraph $(G, c_G) \times (H, c_H)$ of $G \times H$ induced by

$$V((G, c_G) \times (H, c_H)) = \{(u,v) \in V(G \times H) : c_G(u) = c_H(v)\}.$$

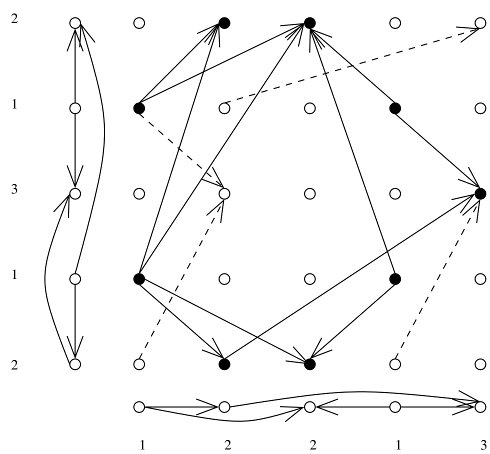

Figure 1: The graph induced by the bold vertices is the fiber product of two 5 cycles with the given colourings. The fiber product is in this case 3-colourable.

Carbonneaux, Gravier, Khelladi and Semri [2] proved that for $n \le 3$, if $\chi(G) = \chi(H) = n$, then $\chi((G, c_G) \times (H, c_H)) = n$. More generally, Khelladi [6] proposed the following.

Conjecture 1.4. [6] *The fiber product of n-chromatic graphs with respect to n-colourings is n-chromatic.*

According to Hedetniemi's Conjecture, for every pair G, H of n-chromatic graphs, there exists an n-chromatic subgraph X of $G \times H$ which admits homomorphisms to both G and H. According to Conjecture 1.4, if G and H are properly n-coloured, then there exists a properly n-coloured subgraph X of $G \times H$ which admits colour-preserving homomorphisms to both G and H.

The main result of this paper is a refutation of Conjecture 1.4.

Theorem 1.5. *For every integer $n \geq 4$, there exist n-chromatic graphs G, H with proper n-colourings c_G, c_H such that $\chi((G, c_G) \times (H, c_H)) < n$.*

Our construction, given in the next section, is based on directed graphs and the directed version of Hedetniemi's conjecture. All the conjectures on the possible structure of n-chromatic proper subgraphs of categorical products of two n-chromatic graphs seem to have been disproved. It seems worthwhile, therefore, to ask the following question about n-critical graphs.

Problem 1.6. *Does there exist an integer $n \geq 5$ and n-chromatic graphs G and H such that $G \times H$ is n-critical?*

2. Proof of Theorem 1.5

If $\vec{G}$ and $\vec{H}$ are directed graphs, their categorical product has vertex-set $V(\vec{G} \times \vec{H}) = V(\vec{G}) \times V(\vec{H})$, and its arcs are the couples $((u, u'), (v, v'))$ such that (u, v) is an arc of $\vec{G}$ and (u', v') is an arc of $\vec{H}$. The chromatic number of a directed graph $\vec{G}$ is defined to be the chromatic number of the undirected graph G obtained by ignoring the directions of the arcs.

Let P_n be the *directed path* on n vertices, that is, the digraph with vertices $1, 2, \ldots, n$ and arcs $(i, i+1)$, $i = 1, \ldots, n-1$. It is well known that if $\vec{G}$ is an acyclic digraph such that $\chi(\vec{G}) \geq n$, then $\vec{G}$ contains a copy of P_n. Define height$(\vec{G})$ to be the largest n such that $\vec{G}$ contains a copy of P_n. Thus, for an acyclic digraph $\vec{G}$, we have $\chi(\vec{G}) \leq \text{height}(\vec{G})$.

It is well known that Hedetniemi's conjecture fails for directed graphs. The simplest example (from [7]) consists of two tournaments $\vec{G}$ and $\vec{H}$ on the vertex-set $\{1, 2, 3, 4\}$. $\vec{G}$ is the transitive tournament T_4 with arcs (i, j) such that $i < j$, and $\vec{H}$ is obtained from T_4 by replacing $(1, 4)$ by $(4, 1)$. Thus $\chi(\vec{G}) = \chi(\vec{H}) = 4$. A three colouring of $\vec{G} \times \vec{H}$ (with colours $1, 2, 3$) is illustrated in the following table.

$i \backslash j$	1	2	3	4
4	2	3	3	1
3	2	2	3	1
2	1	2	3	3
1	1	2	2	3

Colour of (i, j).

In this example, $\vec{G}$ is acyclic, but $\vec{H}$ is not. It is possible to find examples where both factors are acyclic, by taking products with large enough transitive tournaments. For instance, $\chi(\vec{H} \times T_6) = 4$, thus $\vec{G} \times (\vec{H} \times T_6)$ is a 3-chromatic product of 4-chromatic acyclic factors. ($\vec{H} \times T_5$ is acyclic, but 3-chromatic.) We will show that the height can be controlled as well:

Theorem 2.1. *For every integer $n \geq 4$, there exist digraphs $\vec{G}, \vec{H}$ such that $\chi(\vec{G}) = height(\vec{G}) = n$, $\chi(\vec{H}) = height(\vec{H}) = n$, and $\chi(\vec{G} \times \vec{H}) < n$.*

Proof of Theorem 1.5 using Theorem 2.1. For a given n, let $\vec{G}, \vec{H}$ be the digraphs of the statement of Theorem 2.1, and G, H the graphs obtained by ignoring their orientations. Let $c_G : V(G) \to \{1, \dots, n\}$ and $c_H : V(H) \to \{1, \dots, n\}$ be the proper colourings derived from the orientations of $\vec{G}$ and $\vec{H}$ respectively. (That is, the sources are coloured with colour 1, the sources in what remains are coloured with colour 2, and so on.) Then the vertices of $(G, c_G) \times (H, c_H)$ are vertices of $\vec{G} \times \vec{H}$. For every edge $[(u, v), (u', v')]$ of $(G, c_G) \times (H, c_H)$ with say $c_G(u) = c_H(v) < c_G(u') = c_H(v')$, (u, u') is an arc of $\vec{G}$ and (v, v') is an arc of $\vec{H}$, hence $((u, v), (u', v'))$ is an arc of $\vec{G} \times \vec{H}$. Therefore an orientation of $(G, c_G) \times (H, c_H)$ is a subdigraph of $\vec{G} \times \vec{H}$, thus $\chi((G, c_G) \times (H, c_H)) \leq \chi(\vec{G} \times \vec{H}) < n$. □

3. Proof of Theorem 2.1

We will use two constructions to build witnesses to the validity of Theorem 2.1. The first is the arc graph construction δ applied to transitive tournaments. The vertices of $\delta(T_m)$ are the arcs of T_m, that is, the couples (i, j) such that $1 \leq i < j \leq m$. The arcs of $\delta(T_m)$ are the couples $((i, j), (j, k))$ such that $(i, j), (j, k) \in V(\delta(T_m))$.

The second construction used is exponentiation. Given $\vec{G}$ and $\vec{K}$, the *exponential digraph* $\vec{K}^{\vec{G}}$ is the digraph whose vertices are all functions $f : V(\vec{G}) \to V(\vec{K})$, and whose arcs are the pairs (f, g) of functions such that for each arc (u, v) of $\vec{G}$, $(f(u), g(v))$ is an arc of $\vec{K}$.

Let $n \geq 4$ be an integer. We will show that for $m = m_n$ large enough, the digraphs

$$\begin{aligned} \vec{G} &= \delta(T_m) \times T_n \\ \vec{H} &= K_{n-1}^{\delta(T_m) \times T_n} \times T_n \end{aligned}$$

witness the validity of Theorem 2.1. More precisely they have the following properties:

(i) $\vec{G}$ and $\vec{H}$ are acyclic, and height$(\vec{G}) \leq n$, height$(\vec{H}) \leq n$

(ii) $\chi(\vec{G} \times \vec{H}) \leq n - 1$,

(iii) $\chi(\vec{G}) \geq n$,

(iv) $\chi(\vec{H}) \geq n$.

(Note that (i), (iii) and (iv) imply height$(\vec{G}) = \chi(\vec{G}) = n$ and height$(\vec{H}) = \chi(\vec{H}) = n$.)

(i) follows from the fact that T_n is a factor of both $\vec{G}$ and $\vec{H}$, which implies that they are acyclic and have height (and chromatic number) at most n.

(ii) is a consequence of the basic property of exponentiation (see for instance [8, 9, 11]): There exists a homomorphism of $\vec{A} \times \vec{B}$ to $\vec{C}$ if and only if there exists a homomorphism of $\vec{B}$ to $\vec{C}^{\vec{A}}$. In particular, $\vec{A} \times \vec{C}^{\vec{A}}$ admits a homomorphism to $\vec{C}$. In our case, $\vec{G} \times \vec{H}$ admits a homomorphism to $(\delta(T_m) \times T_n) \times K_{n-1}^{\delta(T_m)\times T_n}$ which admits a homomorphism to K_{n-1}. Therefore $\chi(\vec{G} \times \vec{H}) \leq n-1$.

(iii) is the item for which we need to select m. Since $\chi(T_n) > n-1$, $K_{n-1}^{T_n}$ has no loops, hence it has a finite chromatic number, say k. Put $m = 2^k + 1$. It is well known that $2^k + 1 = \chi(T_m) \leq 2^{\chi(\delta(T_m))}$ hence $\chi(\delta(T_m)) > k = \chi(K_{n-1}^{T_n})$ (see [7, 5]). Therefore there is no homomorphism of $\delta(T_m)$ to $K_{n-1}^{T_n}$, whence $\chi(\vec{G}) = \chi(\delta(T_m) \times T_n) > n-1$.

To prove (iv) we need to show that there is no homomorphism of $\vec{H} = K_{n-1}^{\delta(T_m)\times T_n} \times T_n$ to K_{n-1}, that is, no homomorphism of $K_{n-1}^{\delta(T_m)\times T_n}$ to $K_{n-1}^{T_n}$. We use the fact that "exponentiation reverses arrows", that is, if there is a homomorphism of A to B, then there is a homomorphism of C^B to C^A (see [8, 9, 11]). The first projection is a homomorphism of $\delta(T_m) \times T_n$ to $\delta(T_m)$, hence there is a homomorphism of $K_{n-1}^{\delta(T_m)}$ to $K_{n-1}^{\delta(T_m)\times T_n}$. If there were a homomorphism of $K_{n-1}^{\delta(T_m)\times T_n}$ to $K_{n-1}^{T_n}$, there would then be a homomorphism of $K_{n-1}^{\delta(T_m)}$ to $K_{n-1}^{T_n}$. We show that this is impossible.

Put $A = K_{n-1}^{\delta(T_m)}$ and $B = K_{n-1}^{T_n}$. A homomorphism of A to B would imply the existence of a homomorphism of K_{n-1}^B to K_{n-1}^A. According to [9], there are homomorphisms both ways between K_{n-1}^B and the disjoint union $K_{n-1} \oplus T_n$, and there are homomorphisms both ways between K_{n-1}^A and the disjoint union $K_{n-1} \oplus \delta(T_m)$. We have $\chi(K_{n-1}) < \chi(T_n)$, and $\delta(T_m)$ is triangle-free, hence there is no homomorphism of T_n to $K_{n-1} \oplus \delta(T_m)$. Therefore there is no homomorphism from A to B, which concludes the proof. □

The examples built with this construction are quite large. For $n = 4$, we can use the fact that $k = \chi(K_3^{T_4}) = 6$ (see [1]). This gives $m = 2^k + 1 = 65$, hence $\vec{G} = \delta(T_{65}) \times T_4$ with 16640 vertices, and $\vec{H} = K_3^{\vec{G}} \times T_4$, with $3^{16640} \cdot 4$ vertices.

References

[1] S. Bessy and S. Thomassé, The categorical product of two 5-chromatic digraphs can be 3-chromatic, *Discrete Math.*, **305** (2005), 344–346.

[2] Y. Carbonneaux, S. Gravier, A. Khelladi and A. Semri, Coloring fiber product of graphs, *AKCE Int. J. Graphs Comb.*, **3** (2006), 59–64.

[3] M.El-Zahar and N.Sauer, The chromatic number of the product of two 4-chromatic graphs is 4, *Combinatorica*, **5** (1985), 121–126.

[4] S. H. Hedetniemi, *Homomorphisms of graphs and automata*, University of Michigan Technical Report 03105-44-T, 1966.

[5] P. Hell and J. Nešetřil, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and its Applications, 28. Oxford University Press, Oxford, 2004. xii+244 pp.

[6] A. Khelladi, *Sur une opération binaire de graphes*, Communication to the 32d Week of Science, Damas (Syria, 7-13 November 1992).

[7] S. Poljak and V. Rödl, On the arc-chromatic number of a digraph, *J. Combin. Theory Ser. B*, **31** (1981), 190–198.

[8] N. Sauer, Hedetniemi's conjecture—a survey, *Discrete Math.*, **229** (2001), 261–292.

[9] C. Tardif, Hedetniemi's conjecture and dense Boolean lattices, *Order*, **2** (2011), 181–191.

[10] C. Tardif and X. Zhu, On Hedetniemi's conjecture and the colour template scheme, *Discrete Math.*, **253** (2002), 77–85.

[11] X. Zhu, A survey on Hedetniemi's conjecture, *Taiwanese J. Math.*, **2** (1998), 1–24.