AKCE Int. J. Graphs Comb., 9, No. 1 (2012), pp. 59-70

Signed reinforcement numbers of certain graphs*

Ning Li, Xinmin Hou†, Jing Chen and Jun-Ming Xu
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, China.
e-mail: *lning@mail.ustc.edu.cn, xmhou@ustc.edu.cn, ccxxhh@mail.ustc.edu.cn, xujm@ustc.edu.cn*

Communicated by: T.W. Haynes
Received 07 November 2011; accepted 01 March 2012

Abstract

Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to \{-1, 1\}$ is a signed dominating function of G if, for each vertex of G, the sum of the values of its neighbors and itself is positive. The signed domination number of a graph G, denoted $\gamma_s(G)$, is the minimum value of $\sum_{v\in V(G)} f(v)$ over all the signed dominating functions f of G. The signed reinforcement number of G, *denoted* $R_s(G)$, is defined to be the minimum cardinality $|S|$ of a set S of edges such that $\gamma_s(G+S) < \gamma_s(G)$. In this paper, we initialize the study of signed reinforcement number and determine the exact values of $R_s(G)$ for several classes of graphs.

Keywords: Signed domination, signed reinforcement number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of $v \in V(G)$ are defined by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N_G[v] = N_G(v)\cup\{v\}$ and $d_G(v) = |N_G(v)|$, respectively. For $S \subseteq V(G)$, $N_G(S)$ is defined to be the union of the open neighborhoods $N_G(v)$ for all $v \in S$ and $N_G[S] = N_G(S) \cup S$. Let $\Delta(G)$ denote the maximum degree of a graph G. A vertex of degree one in G is called a leaf; a support vertex of G is a vertex adjacent with a leaf of G. Let $L(G)$ and $S(G)$ denote the set of leaves of G and the set of support vertices of G, respectively. For two sets $A, B \subseteq V(G)$, let $E(A, B) = \{e = xy \mid x \in A, y \in B\}$ and $e(A, B) = |E(A, B)|$.

Let $G = (V, E)$ be a graph and $f : V \to R$ is a real-valued function on V. The weight of f is $\omega(f) = \sum_{v\in V} f(v)$. For $S \subseteq V$, define $f(S) = \sum_{v\in S} f(v)$. Then $\omega(f) = f(V)$.

---

*The work was supported by NNSF of China and the Fundamental Research Funds for the Central Universities.

†corresponding author

For any $v \in V$, let $f[v] = f(N[v])$ for notation convenience. A function $f : V \rightarrow \{-1, 1\}$ is called a signed dominating function (abbreviated by SDF) if $f[v] \geq 1$ for all $v \in V$. The *signed domination number* of G is $\gamma_s(G) = \min\{\omega(f) \mid f \text{ is a SDF of } G\}$. A $\gamma_s(G)$-function is a signed dominating function of G of weight $\gamma_s(G)$. Signed domination was first introduced by Dunbar et al. in [4] and further studied in [1, 3, 6, 7, 9, 14, 10, 11, 12, 13, 15].

The reinforcement number of a graph G is a measurement of the stability of the domination in G. The reinforcement number of a graph G is the smallest number of edges which must be added to G to decrease the domination number of G (the classic domination number of a graph G is the minimum cardinality of a subset D of $V(G)$ such that for each $v \in V(G)$, $N[v] \cap D \neq \emptyset$). The definition was first introduced by Kok and Mynhardt [8]. During the past twenty years, the reinforcement number associated with domination parameters were studied in literatures, for example, Ghoshal et al.[5] defined and studied the reinforcement number associated with the strong domination number; Gayla et al.[2] studied the reinforcement number associated with the fractional domination number.

In this paper, we define the signed reinforcement number of a graph G, *denoted* $R_s(G)$, to be the minimum cardinality of a set S of edges in the complement graph G^c of G such that $\gamma_s(G + S) < \gamma_s(G)$. A minimum edge set $S \subseteq E(G^c)$ with $\gamma_s(G + S) < \gamma_s(G)$ is called a signed reinforcement set of G. Note that the signed reinforcement set of a graph G maybe doesn't exist, for example, for K_n, the complete graph on n vertices or C_4, the cycle on 4 vertices. So if the signed reinforcement set of a graph G doesn't exist, we define $R_s(G) = 0$.

The paper is organized as follows. Section 2 gives some lemmas about signed domination numbers and signed reinforcement numbers. Sections 3 and 4 determine the exact values of the signed reinforcement numbers of paths, cycles and wheels. Section 5 gives a sharp bound of the signed reinforcement number of trees.

2. Lemmas

In this section, we will give some useful lemmas about signed dominating functions of a graph G. Let K_n, P_n and C_n denote a complete graph, a path and a cycle on n vertices, respectively. The following lemmas are given in [4] and the proof of them can be found in [4].

Lemma 2.1. [4] *A signed dominating function f on a graph G is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N[v]$ with $f[u] \in \{1, 2\}$.*

Lemma 2.2. [4] *If f is a signed dominating function of a graph G, then $f(v) = 1$ for any $v \in L(G) \cup S(G)$.*

Lemma 2.3. [4] *Let G be a graph on n vertices. Then $\gamma_s(G) = n$ if and only if $V(G) = L(G) \cup S(G)$.*

Lemma 2.4. [4] *If G has more than three vertices and maximum degree $\Delta \leq 3$, then $\gamma_s(G) \geq \frac{n}{3}$.*

The following lemma gives a lower bound for the signed domination number of a graph G with precisely one vertex with maximum degree four.

Lemma 2.5. *Let G be a graph with order n and maximum degree four. If G has precisely one vertex with maximum degree four, then $\gamma_s(G) \geq \frac{n-2}{3}$.*

Proof. Let f be a γ_s-function and let P and M be the reverse images of +1 and -1 under f. Then $|P| + |M| = n$ and $\gamma_s(G) = |P| - |M|$.

If $M = \emptyset$, then $\gamma_s(G) = n > \frac{n-2}{3}$.

If $M \neq \emptyset$, we evaluate the number, $e(M, P)$, of edges between P and M in G.

For any $v \in M$, to guarantee $f[v] \geq 1$, there exist at least two edges from v to P, which means that $e(M, P) \geq 2|M|$.

On the other hand, for each $v \in P$, to guarantee $f[v] \geq 1$, $|N(v) \cap M| \leq |N(v) \cap P|$. Hence there are at most $\lfloor \frac{d(v)}{2} \rfloor$ edges from v to M. Since G has precisely one vertex with maximum degree four, $e(P, M) \leq |P| - 1 + 2 = |P| + 1$.

Hence, $2|M| \leq e(M, P) \leq |P| + 1$. Combine with $|P| + |M| = n$, we have $|M| \leq \frac{n+1}{3}$ and $|P| \geq \frac{2n-1}{3}$. So, $\gamma_s(G) = |P| - |M| \geq \frac{2n-1}{3} - \frac{n+1}{3} = \frac{n-2}{3}$. □

The signed domination numbers of paths, cycles and stars were given in [4].

Lemma 2.6. [4]

1. $\gamma_s(K_{1,n-1}) = n$, $n \geq 2$;
2. $\gamma_s(P_n) = n - 2\lfloor \frac{n-2}{3} \rfloor$, $n \geq 2$;
3. $\gamma_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$, $n \geq 3$.

Lemma 2.7. *Let G be a connected graph with $|V(G)| \geq 3$. If $\gamma_s(G) = |V(G)|$, then $R_s(G) = 1$.*

Proof. Since $\gamma_s(G) = |V(G)|$, by Lemma 2.3, $V(G) = L(G) \cup S(G)$. Let f be a γ_s-function. Since $\gamma_s(G) = |V(G)|$, $f \equiv 1$. Since $|V(G)| \geq 3$, $|L(G)| \geq 2$. Let $u, v \in L(G)$ and w be the support vertex of u. Since $f \equiv 1$, $f[w] \geq 3$. Then if we replace the value 1 by -1 on u and adding the edge uv to G, then the reduced function is a SDF of $G + uv$ with weight $|V(G)| - 2 < \gamma_s(G)$. So $R_s(G) = 1$. □

Lemma 2.8. *For any graph G, if $\gamma_s(G + A) < \gamma_s(G)$ for some set $A \subseteq E(G^c)$, then $\gamma_s(G + A) \leq \gamma_s(G) - 2$.*

Proof. Let f and g be minimum signed dominating functions of $G + A$ and G, respectively, and let $f^{-1}(a)$ and $g^{-1}(a)$ denote the reversed imagines of a under f and g. Since $\gamma_s(G + A) < \gamma_s(G)$, $|f^{-1}(1)| \leq |g^{-1}(1)| - 1$ (equivalently, $|f^{-1}(-1)| \geq |g^{-1}(-1)| + 1$). Hence $\gamma_s(G + A) = |f^{-1}(1)| - |f^{-1}(-1)| \leq |g^{-1}(1)| - |g^{-1}(-1)| - 2 = \gamma_s(G) - 2$. □

3. The signed reinforcement numbers of stars, paths and cycles

Since $V(K_{1,n-1}) = L(K_{1,n-1}) \cup S(K_{1,n-1})$, by Lemmas 2.3 and 2.7, $R_s(K_{1,n-1}) = 1$ if $n \geq 3$. Hence, we have the following observation.

Observation 3.1. *Let $n \geq 3$. Then $R_s(K_{1,n-1}) = 1$.*

Theorem 3.2. *For $n \geq 3$,*

$$R_s(P_n) = \begin{cases} 2, & n \equiv 2 \pmod 3 \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. Denote $V(P_n) = \{v_1, v_2, \cdots, v_n\}$.

If $n = 3k$ or $3k + 1$ for some integer $k(\geq 1)$, then

$$\gamma_s(P_{3k} + v_1 v_{3k}) = \gamma_s(C_{3k}) = k < k + 2 = \gamma_s(P_{3k})$$

and

$$\gamma_s(P_{3k+1} + v_1 v_{3k+1}) = \gamma_s(C_{3k+1}) = k + 1 < k + 3 = \gamma_s(P_{3k+1}).$$

This implies that $R_s(P_n) = 1$ if $n \neq 2(mod 3)$.

If $n = 3k + 2$ for some integer $k \geq 1$, let G be the graph obtained from P_{3k+2} by adding two edges $v_1 v_3, v_3 v_{3k+2}$. Now, define a function f as follows:

$$f(v_i) = \begin{cases} -1, & i \equiv 1 \pmod 3 \\ 1, & \text{otherwise} \end{cases}$$

It is an easy task to check that$f[v_i] = 1$ for every $i \in [1, 3k + 2]$. So f is a SDF of G. Hence, $\gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(P_{3k+2})$. Therefore, $R_s(P_{3k+2}) \leq 2$.

Now we show that $R_s(P_{3k+2}) = 2$. If there exists some edge $e \notin E(P_{3k+2})$ such that $\gamma_s(P_{3k+2} + e) < \gamma_s(P_{3k+2})$, then, by Lemma 2.8, $\gamma_s(P_{3k+2} + e) \leq \gamma_s(P_{3k+2}) - 2 = k$. Since $\Delta(P_{3k+2} + e) \leq 3$, by Lemma 2.4, $\gamma_s(P_{3k+2} + e) \geq \lceil \frac{3k+2}{3} \rceil = k + 1 > k \geq \gamma_s(P_{3k+2} + e)$, a contradiction. □

Lemma 3.3. *Let $n \geq 3$ and $n \equiv 0$ or $1 \pmod 3$. Then*

$$R_s(C_n) = \begin{cases} 0, & n = 3, 4 \\ 3, & n \geq 6. \end{cases}$$

Proof. Denote $V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ and $E(C_n) = \{v_i v_{i+1} | \ i = 0, 1, \cdots, n-1\}$, where the "+" is under modulo n. If $n = 3$, then $C_n = K_3$ and $R_s(C_n) = R_s(K_3) = 0$. If $n = 4$, we can check that $\gamma_s(C_4) = 2 = \gamma_s(C_4 + v_1 v_3) = \gamma_s(C_4 + v_0 v_2) = \gamma_s(C_4 + \{v_1 v_3, v_0 v_2\})$ and hence $R_s(C_4) = 0$.

If $n \geq 6$, let G be the graph obtained from C_n by adding three edges v_1v_3, v_1v_5 and v_3v_5 and define a function $f : V(G) \to \{-1, 1\}$ by

$$f(v_i) = \begin{cases} -1, & i = 2, 4 \text{ or } 3j \text{ for } j \in [2, \lfloor \frac{n}{3} \rfloor] \\ 1, & \text{otherwise.} \end{cases}$$

It is an easy task to check that $f[v_i] \geq 1$ for any $i \in [1, n]$. So f is a SDF of G and hence $\gamma_s(G) \leq f(V(G)) = n - 2(\lfloor \frac{n}{3} \rfloor + 1) = n - 2\lfloor \frac{n}{3} \rfloor - 2 < n - 2\lfloor \frac{n}{3} \rfloor = \gamma_s(C_n)$ (Lemma 2.6 (3)). So we have $R_s(C_n) \leq 3$.

Next we will show that $R_s(C_n) \geq 3$ and so the result follows. Suppose to the contrary that there exist two edges e_1,$e_2 \notin E(C_n)$ such that $\gamma_s(C_n + \{e_1, e_2\}) < n - 2\lfloor \frac{n}{3} \rfloor = \gamma_s(C_n)$. By Lemma 2.8, $\gamma_s(C_n + \{e_1, e_2\}) \leq \gamma_s(C_n) - 2$.

If e_1, e_2 are independent, then $\Delta(C_n + \{e_1, e_2\}) \leq 3$. By Lemma 2.4, $\gamma_s(C_n + \{e_1, e_2\}) \geq \lceil \frac{n}{3} \rceil = n - 2\lfloor \frac{n}{3} \rfloor = \gamma_s(C_n)$, a contradiction.

If e_1, e_2 have a common end, then $C_n + \{e_1, e_2\}$ has precisely one vertex with maximum degree four. By Lemma 2.5, $\gamma_s(C_n + \{e_1, e_2\}) \geq \lceil \frac{n-2}{3} \rceil \geq n - 2\lfloor \frac{n}{3} \rfloor - 1 = \gamma_s(C_n) - 1$, a contradiction too. □

Lemma 3.4. *If* $n \equiv 2 \pmod 3$ *and* $n \geq 5$*, then* $R_s(C_n) = 2$.

Proof. Let $V(C_n)$ and $E(C_n)$ be defined the same as in the former proof and let G be the graph obtained by adding two edges v_1v_3 and v_3v_5. Suppose $n = 3k + 2$ $(k \geq 1)$. Define a function $f : V(G) \to \{-1, 1\}$ by

$$f(v_i) = \begin{cases} -1, & i = 2 \text{ or } 3j + 1 \text{ for } j \in [1, k] \\ 1, & \text{otherwise.} \end{cases}$$

It is an easy task to check that $f[v_i] \geq 1$ for any $i \in [1, n]$. So f is a SDF of G and $\gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(C_n)$. Hence $R_s(C_n) \leq 2$.

If we can show that $R_s(C_n) \geq 2$, then the result follows. Suppose that there exists some edge $e \notin E(C_n)$ such that $\gamma_s(C_n + e) < \gamma_s(C_n)$. By Lemma 2.8, $\gamma_s(C_n + e) \leq \gamma_s(C_n) - 2 = k$. Since $\Delta(C_n + e) = 3$, $\gamma_s(C_n + e) \geq \lceil \frac{n}{3} \rceil = k + 1$ by Lemma 2.4, a contradiction with $\gamma_s(C_n + e) \leq k$. □

From the above two lemmas, we have

Theorem 3.5. *Let* $n \geq 3$*. Then*

$$R_s(C_n) = \begin{cases} 0, & n = 3, 4 \\ 3, & n \equiv 0 \text{ or } 1 \pmod 3 \text{ and } n \geq 6 \\ 2, & n \equiv 2 \pmod 3. \end{cases}$$

4. Wheels

A wheel is a graph obtained from a cycle by adding a new vertex such that it is adjacent with each vertex of the cycle. Let $W_n = \{w\} \vee C_{n-1}$ denote a wheel obtained from a cycle C_{n-1} and a new vertex w, called the central vertex of W_n. In the following, we denote $V(C_{n-1}) = \{v_0, v_1, \cdots, v_{n-2}\}$ and $E(W_n) = \{wv_i, v_iv_{i+1}, i = 0, 1, \cdots, n-2\}$, where the sum is taken modulo $n-1$.

First we determine the signed domination number of W_n.

Lemma 4.1. *For $n \geq 4$, $\gamma_s(W_n) = n - 2\lfloor \frac{n-1}{3} \rfloor$.*

Proof. Since we can extend a SDF of C_{n-1} to be a SDF of W_n by assigning 1 to the central vertex w, $\gamma_s(W_n) \leq \gamma_s(C_{n-1}) + 1 = n - 1 - 2\lfloor \frac{n-1}{3} \rfloor + 1 = n - 2\lfloor \frac{n-1}{3} \rfloor$.

In the following, we show that $\gamma_s(W_n) \geq n - 2\lfloor \frac{n-1}{3} \rfloor$. Let f be a minimum SDF of W_n and let P and M be the set of reverse imagines of 1 and -1 under f, respectively. We claim that $f(w) = 1$, equivalently, $w \in P$. If $f(w) = -1$, to guarantee $f[v_i] \geq 1$ for any $i = 0, \cdots, n-2$, $f(v_i) = 1$ since $d(v_i) = 3$. This means that $\gamma_s(W_n) = n - 1 > n - 2\lfloor \frac{n-1}{3} \rfloor$, a contradiction.

Since $f[v_i] = f(w) + f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 1$, $f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 0$. Hence at most one of three consecutive vertices on C_{n-1} is assigned -1 by f. This implies that $|M| \leq \frac{n-1}{3}$. So $\gamma_s(W_n) = n - 2|M| \geq n - 2\lfloor \frac{n-1}{3} \rfloor$. □

Theorem 4.2.

1. $R_s(W_4) = 0$.
2. *If $n \geq 5$,*

$$R_s(W_n) = \begin{cases} 2, & n = 1(mod3) \\ 1, & otherwise \end{cases} .$$

Proof. (1) It follows directly from $W_4 = K_4$ and $R_s(K_n) = 0$ for any $n \geq 2$.

(2) If $n = 3k$ ($k \geq 2$) or $n = 3k+2$ ($k \geq 1$), then, by Lemma 4.1, $\gamma_s(W_n) = n - 2\lfloor \frac{n-1}{3} \rfloor = k+2$. Now, we add an edge v_0v_2 to W_n and define a function $g : V(W_n + v_0v_2) \to \{-1, 1\}$ as follows:

$$g(x) = \begin{cases} -1, & \text{if } x = v_i \text{ and } i = 1, n-2 \text{ or } 3j \text{ for } j \in [1, \lceil \frac{n}{3} \rceil - 2] \\ 1, & \text{otherwise} \end{cases} .$$

It is an easy task to check that g is a SDF of $W_n + v_0v_2$. Hence $\gamma_s(W_n + v_0v_2) \leq g(V(W_n)) = n - 2\lceil \frac{n}{3} \rceil = k < k+2 = \gamma_s(W_{3k})$. So $R_s(W_n) = 1$.

If $n = 3k+1$ ($k \geq 2$), then, by Lemma 4.1, $\gamma_s(W_{3k+1}) = 3k + 1 - 2\lfloor \frac{3k+1-1}{3} \rfloor = k+1$. Then we can add two edges v_0v_2, v_2v_4 to W_n and define a SDF g of $W_n + \{v_0v_2, v_2v_4\}$ as follows:

$$g(x) = \begin{cases} -1, & \text{if } x = v_i \text{ and } i = 1, 3 \text{ or } 3i-1 \text{ for } i \in [2,k] \\ 1, & otherwise \end{cases}.$$

Hence

$$\gamma_s(W_{3k+1} + \{v_0v_2, v_2v_4\}) \le g(V(W_n)) = 3k + 1 - 2(k+1) = k - 1 < k + 1 = \gamma_s(W_{3k+1}).$$

So $R_s(W_{3k+1}) \le 2$.

In the following, we will prove that $R_s(W_{3k+1}) \ge 2$. Suppose to the contrary that there exists an edge $e \notin E(W_n)$ such that $\gamma_s(W_{3k+1} + e) < \gamma_s(W_{3k+1}) = k + 1$.

Let ϕ be a minimum SDF of $W_{3k+1} + e$ and let P and M be the reverse imagines of 1 and -1 under ϕ, respectively. Then,

$$\begin{cases} |P| + |M| = 3k + 1 \\ |P| - |M| = \gamma_s(W_{3k+1} + e) \le k \end{cases}.$$

Since $|M|$ and $|P|$ are integers, the equation array implies that

$$\begin{cases} |M| \ge k + 1 \\ |P| \le 2k \end{cases}.$$

With a same reason with $f(w) = 1$ in the proof of Lemma 4.1, $\phi(w) = 1$. Then $M \subseteq V(C_{3k})$. Since $|M| \ge k + 1$, there are three consecutive vertices v_{i-1}, v_i, v_{i+1} on C_{3k} such that two of them are in M.

If the two members of $\{v_{i-1}, v_i, v_{i+1}\} \cap M$ are consecutive on C_{3k}, without loss of generality, suppose $v_{i-1}, v_i \in M$. Then, to guarantee that $\phi[v_{i-1}] \ge 1, \phi[v_i] \ge 1$, $d(v_{i-1}) \ge 4$ and $d(v_i) \ge 4$. This is impossible since v_{i-1}, v_i can not be the two ends of the new adding edge e. Hence we must have $v_{i-1}, v_{i+1} \in M$.

To guarantee $\phi[v_i] \ge 1$, $d(v_i) = 4$ and $\phi(v_i) = 1$. This means that v_i must be an end of e. Suppose $e = v_iv_m$. Then $\phi(v_m) = 1$. Now we compute the number of edges between M and $P - \{w\}$ with two methods. Let $P' = P - \{w\}$.

Since, for each $x \in M$, $d(x) = 3$ and $\phi[x] = -1 + \phi(w) + \phi(N(x) \setminus \{w\}) \ge 1$, $\phi(N(x) \setminus \{w\}) \ge 1$. So, there are two edges from x to the vertices in P', this means $e(x, P') = 2$. Hence $e(M, P') = 2|M| \ge 2(k + 1)$.

Since, for each $x \in P - \{w, v_i, v_m\})$, $d(x) = 3$ and $\phi[x] = 1 + 1 + \phi(N(x) \setminus \{w\}) \ge 1$, $\phi(N(x) \setminus \{w\}) \ge -1$. Hence there is at most one edge from x to the vertices in M, which means that $e(x, M) \le 1$ for each $x \in V(P - \{w, v_i, v_m\})$. For v_i and v_m, there are at most 2 edges from v_i or v_m to vertices in M. So,

$$e(P', M) \le |P - \{w, v_i, v_m\}| + 4 \le 2k - 3 + 4 = 2k + 1 < 2(k + 1) \le e(M, P'),$$

a contradiction. □

5. Trees

Lemma 5.1. *For any tree T with order $n \geq 3$, $R_s(T) \leq 3$.*

Proof. If $\gamma_s(T) = n$, then $R_s(T) = 1 < 3$ by Lemma 2.7.

Now suppose $\gamma_s(G) < n$. Then $T \neq K_{1,n-1}$. Hence there exist two leaves u_1, v_1 such that they have two different support vertices u_2 and v_2, respectively.

If $L(T) = \{u_1, v_1\}$, then $T = P_n$ and so $R_s(T) \leq 2$ by Theorem 3.2.

If $L(T) \neq \{u_1, v_1\}$, let w_1 be another leaf of T. Then there is at least one of u_2, v_2 which is not adjacent with w_1. Without loss of generality, assume $u_2w_1 \notin E(T)$. Let f be a minimum SDF of T. By Lemma 2.2, $f(u_i) = f(v_i) = 1, i = 1, 2$ and $f(w_1) = 1$. Let $S = \{u_1v_1, u_2w_1\}$ if $u_2v_2 \in E(G)$ and $S = \{u_1v_1, u_2v_2, u_2w_1\}$ if $u_2v_2 \notin E(G)$. We can easily modify f to be a SDF g of $T + S$ as follows.

$$g(x) = \begin{cases} -1, & x = u_1 \\ f(x), & x \in V(T) - \{u_1\} \end{cases} .$$

Then $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2 < \gamma_s(T)$ implies that $R_s(T) \leq 3$. □

In fact, the upper bound of the signed reinforcement number of trees given here is not sharp. In the following, we will give a sharp bound for $R_s(T)$.

Lemma 5.2. *Let f be a minimum SDF of a tree T. If there exists a support vertex v with $f[v] \geq 3$, then $R_s(T) = 1$.*

Proof. Let u, w be two leaves of T such that at least one of them is adjacent with v in T. Then uw is the desired edge to guarantee that $\gamma_s(T + uw) < \gamma_s(T)$. □

Lemma 5.3. *Let f be a minimum SDF of a tree T. If there exists a support vertex v with $f[v] \geq 2$, then $R_s(T) \leq 2$.*

Proof. If T is a star, then the result is clearly true. Now suppose T is not a star. Then we can choose two leaves u, w of T such that $uv \in E(T)$ and $wv \notin E(T)$. Hence uw, vw are two edges to guarantee that $\gamma_s(T + \{uw, vw\}) < \gamma_s(T)$. So, $R_s(T) \leq 2$. □

Lemma 5.4. *Let T' be a tree obtained from a tree T ($|V(T)| \geq 3$) by adding an edge joining a leaf of T with a leaf of a path P_3. Then $R_s(T') \leq R_s(T)$.*

Proof. Let v be a leaf of T and let u be its support vertex. Let $P_3 = x_1x_2x_3$ and let T' be a tree obtained from $T \cup P_3$ by adding an edge x_3v. It is an easy task to check that $\gamma_s(T') = \gamma_s(T) + 1$. Now suppose $R_s(T) = r$ and S is a set of edges with $|S| = r$ such that $\gamma_s(T + S) \leq \gamma_s(T) - 2$ (by Lemma 2.8). By Lemma 5.1, $r \leq 3$.

Let f be a minimum SDF of $T + S$. Then $f(V(T)) = \gamma_s(T + S)$.

If v is not incident with any edge in S, then v is a leaf of $T+S$, too. Hence $f(v) = f(u) = 1$. We can easily extend f to be a SDF, say g, of $T'+S$ by defining $g(x_1) = g(x_2) = 1$ and $g(x_3) = -1$ and $g(x) = f(x)$ for the other vertices. So $\gamma_s(T'+S) \le \gamma_s(T+S)+1 \le \gamma_s(T)-2+1 = \gamma_s(T')-2$. This implies that $R_s(T^{'}) \le |S| = r = R_s(T)$.

Now we suppose that v is incident with some edges, denoted $vu_1, \cdots, vu_t$, in S.

If $f[v] \ge 2$, then we can extend f to be a SDF of $T'+S$ the same as the above case and so the result is valid. So we assume that $f[v] = 1$ in the following.

Case 1. $f(v) = 1$.

If $f(u) = 1$, then $f(u_1) + \cdots + f(u_t) = -1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. Then we can define a SDF g of $T'+S'$ as follows:

$$g(x) = \begin{cases} -1, & x = x_3 \\ 1, & x = x_1, x_2 \\ f(x), & x \in V(T) \end{cases}.$$

So $\gamma_s(T^{'}+S^{'}) \le g(V(T^{'}+S^{'})) \le \gamma_s(T+S)+1 \le \gamma_s(T)-1 = \gamma_s(T')-2$. This implies that $R_s(T^{'}) \le |S'| = |S| = R_s(T)$.

If $f(u) = -1$, then $f(u_1) + \cdots + f(u_t) = 1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. We also can define a SDF g of $T'+S'$ as follows:

$$g(x) = \begin{cases} -1, & x = x_2 \\ 1, & x = x_1, x_3 \\ f(x), & x \in V(T) \end{cases}.$$

So $\gamma_s(T^{'}+S^{'}) \le g(V(T^{'}+S^{'})) \le \gamma_s(T+S)+1 \le \gamma_s(T)-1 = \gamma_s(T')-2$ implies that $R_s(T^{'}) \le |S'| = |S| = R_s(T)$.

Case 2. $f(v) = -1$.

If $f(u) = 1$, then $f(u_1) + \cdots + f(u_t) = 1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. Then we can extend f to be a SDF g of $T'+S'$ the same as the case $f(v) = 1$ and $f(u) = -1$ and so the result is valid.

If $f(u) = -1$, then $f(u_1) + \cdots + f(u_t) \ge 3$. Since $t \le r \le 3$, $t = 3$ and $f(u_1) = f(u_2) = f(u_3) = 1$. Let $S' = (S - \{vu_1\}) \cup \{x_1u_1\}$ and define

$$g(x) = \begin{cases} -1, & x = x_2 \\ 1, & x = x_1, x_3 \\ f(x), & x \in V(T) \end{cases}.$$

Then g is a SDF of $T'+S'$ and $\gamma_s(T^{'}+S^{'}) \le g(V(T^{'}+S^{'})) \le \gamma_s(T+S)+1 \le \gamma_s(T)-1 = \gamma_s(T')-2$. This also implies that $R_s(T^{'}) \le |S'| = |S| = R_s(T)$. □

Theorem 5.5. *For any tree T of order $n \ge 2$, $R_s(T) \le 2$.*

Proof. We prove the result by induction on the order of T. Since the result is true for $T = K_2$, we assume that $n \geq 3$. If $n = 3$, then, by Theorem 3.2, $R_s(T) = 1$ and the result is true. Now assume that $n \geq 4$ and the result is true for any tree with order less than n. Let T be a tree with $|V(T)| = n$ and let $f : V(T) \to \{-1, 1\}$ be a minimum SDF of T. Then $f(V(T)) = \gamma_s(T)$ and $f(v) = 1$ for any $v \in L(T) \cup S(T)$ by Lemma 2.2.

Let $P_m = v_1v_2 \cdots v_m$ be a longest path of T.

If $d(v_2) \geq 3$, then there are at least two leaves adjacent with v_2 since P_m is a longest path of T. Since $f(v_2) = 1$, $f[v_2] \geq 3 - 1 = 2$. By Lemma 5.3, $R_s(T) \leq 2$ and so the result is true. Hence, in the following, we suppose $d(v_2) = 2$.

Case 1. $d(v_3) \geq 3$.

Case 1.1. If v_3 is adjacent with a leaf x, then $f(x) = f(v_3) = 1$. So $f[v_2] \geq 3$. By Lemma 5.2, $R_s(T) = 1$.

Case 1.2. If v_3 is not adjacent with any leaf of T, since P_m is a longest path of T, each neighbor of v_3 other than v_4 is a support vertex of T. Since $d(v_2) = 2$, we can assume that each component of $T - \{v_3\}$ not containing v_4 is isomorphic to K_2. If $f(v_3) = 1$, then $f[v_2] \geq 3$. By Lemma 5.2, $R_s(T) = 1$. Now we assume $f(v_3) = -1$.

Let y_1y_2 be a component of $T - \{v_3\}$ other than v_1v_2 with $y_2v_3 \in E(T)$. Let $S = \{v_1v_3, y_1v_3\}$. Define a function $g : V(T + S) \to \{-1, 1\}$ as follows:

$$g(x) = \begin{cases} -1, & x = y_1, v_1 \\ 1, & x = v_3 \\ f(x), & otherwise \end{cases}.$$

It is an easy task to check that $g[x] \geq 1$ for any vertex $x \in V(T + S)$ and hence g is a SDF of $T + S$. So $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2$ which implies that $R_s(T) \leq |S| = 2$.

Case 2. $d(v_3) = 2$.

Case 2.1. If $f(v_3) = 1$, then $f[v_2] \geq 3$ and so $R_s(T) = 1$ by Lemma 5.2.

Case 2.2. If $f(v_3) = -1$, then, to guarantee $f[v_3] \geq 1$, $f(v_4)$ must be 1.

If $d(v_4) = 2$, then, to guarantee $f[v_4] \geq 1$, $f(v_5) = 1$. Let $T' = T - \{v_1, v_2, v_3\}$. By the inductive hypothesis, $R_s(T') \leq 2$. Since $\{v_1, v_2, v_3\}$ induce a path P_3, by Lemma 5.4, $R_s(T) = R_s(T' + P_3) \leq R_s(T') \leq 2$.

Now assume that $d(v_4) \geq 3$.

If v_4 is a support vertex and w is a leaf adjacent with v_4, then $f(w) = f(v_4) = 1$. Let $S = \{v_1v_3, wv_3\}$. We can define a SDF g of $T + S$ as follows:

$$g(x) = \begin{cases} -1, & x = w, v_1 \\ 1, & x = v_3 \\ f(x), & otherwise \end{cases}.$$

So $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2$ implying that $R_s(T) \leq 2$.

If v_4 is adjacent with a support vertex y such that $N(y) - \{v_4\}$ are leaves of T, then $f(y) = 1$. Since $f(v_4) = 1$ and the value of any leaf assigned by f is 1, $f[y] \geq 3$. By Lemma 5.2, we have $R_s(T) = 1$.

By the above proofs, we can assume that: (i) each component of $T - \{v_4\}$ not containing v_5 is isomorphic to P_3 with an end adjacent with v_4; (ii) the value of the vertex adjacent with v_4 assigned by f is -1. By this assumption, to guarantee $f[v_4] \geq 1$, there is exactly one such component, that means $d(v_4) = 2$, contradicts with the assumption $d(v_4) \geq 3$. □

Remark 5.6. *The upper bound $R_s(T) \leq 2$ is sharp since $R(P_{3k+2}) = 2, k \geq 1$.*

References

[1] W. Chen and E. Song, Lower bounds on several versions of signed domination number, *Discrete Math.*, **308**(10) (2008), 1837-1846.

[2] G. S. Domke and R. C. Laskar, The bondage and reinforcement number of γ_f for some graphs, *Discrete Math.*, **167/168** (1997), 249-259.

[3] J. E. Dunbar, T. W. Haynes, U. Teschner and L. Volkmann, *Bondage, insensitivity, and reinforcement*, In: T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Editors, Domination in Graphs: Advanced Topics, Marcel Dekker, New York (1998), pp. 471-489.

[4] J. E. Dunbar, S. T. Hedetniemi, M. A. Henning and P. J. Slater, *Signed domination in graphs*, Graph Theory, Combinatorics, and Applications, John Wiley&Sons, Inc. 1 (1995), 311-322.

[5] J. Ghoshal, R. C. Laskar, D. Pillon and C. Wallils, Strong bondage and strong reinforcement numbers of graphs, *Congr. Numer.*, **108** (1995), 33-42.

[6] R. Hass and T. B. Wexler, Signed domination numbers of a graph and its complement, *Discrete Math.*, **283**(1-3) (2004), 87-92.

[7] J. Huang, J. Wang and J.-M. Xu, Reinforcement numbers of digraphs, *Discrete Appl. Math.*, **157**(8) (2009),1938-1946.

[8] J. Kok and C. M. Mynhardt, Reinforcement in graphs, *Congr. Numer.*, **79** (1990), 225-231.

[9] X. Lu, Total signed domination numbers of graphs, *Int. J. Pure Appl. Math.*, **29**(3) (2006), 281-288

[10] X. Lu, A lower bound on the total signed domination numbers of graphs, *Sci. China Ser.A* , **50**(8) (2007), 1157-1162.

[11] S. M. Sheikholeslami, Forcing signed domination number of graphs, *Matematicki Vesnic*, **59** (2007), 171-179.

[12] M. Y. Sohn, J. Lee and Y. S. Kwon, Lower bounds of signed domination number of a graph, *Bull. Korean Math. Soc.*, **41**(1) (2004),181-188.

[13] H. Tang and Y. Chen, Upper signed domination number, *Discrete Math.*, **308**(15) (2008), 3416-3419.

[14] B. Zelinka, Signed domination number of directed graphs, *Czechoslovak Math. J.*, **55**(130) (2005),479-482.

[15] Z. Zhang, B. Xu, Y. Li and L. Liu, A note on the lower bounds of signed domination number of a graph, *Discrete Math.*, **195** (1999), 295-298.