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A corrected version of Meyniel's conjecture

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Abstract

Since its definition in 1944, the concept of kernel of a digraph has been thoroughly studied. It has many applications in game theory, mathematical logic and other branches of mathematics. In this paper we establish a bijection between digraphs and square hypergraphs with a (fixed) system of distinct representatives, which allows us to obtain new results on kernels of digraphs. Among them, we prove a modification of a well-known conjecture by Meyniel.

Keywords: Kernel, square hypergraph, pseudodiagonal.

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1. Introduction

The concept of kernel was introduced in 1944 by Von Neumann and Morgenstern [15]. It has many applications in game theory, logic, code theory and other branches of mathematics. They proved [15] that every acyclic digraph has a unique kernel, and in 1953 Richardson [16] proved that every digraph without odd directed cycles is kernel-perfect. In 1980 Meyniel [5] conjectured that if D is a digraph such that every odd directed cycle has at least two pseudodiagonals, then D has a kernel. This conjecture has been source of many questions and results, although it was disproved in 1982 by Galeana-Sánchez [8].

Since then, much of the research on kernels involved imposing various conditions on the pseudodiagonals of odd directed cycles. In 1980 Duchet [4] proved that if D is a digraph such that every odd directed cycle of D has at least two symmetric arcs, then

D has a kernel. Four years later Galeana-Sánchez and Neumann-Lara [10] showed that if D is a digraph such that every odd directed cycle has at least two pseudodiagonals incident into consecutive vertices, then D has a kernel. In 1986 Galeana-Sánchez [9] proved that if D is a triangle-free digraph such that every odd directed cycle of D has at least two pseudodiagonals of the form (x_i, x_{i+2}), then D has a kernel, and the following year Duchet [6] extended the result from triangle-free digraphs to digraphs in which every triangle is symmetric. Also in 1986, Maffray [13] showed that Meyniel's conjecture holds for so-called i-triangulated digraphs, and in 1992 the same author [14] proved that it also holds for every orientation of the line-graph of any given graph.

In this paper we relate certain transversals of square (multi-)hypergraphs with kernels of digraphs, which allows us to obtain several new sufficient conditions for the existence of kernels in digraphs. In particular, we show that a modification of Meyniel's conjecture holds. For general concepts about digraphs and hypergraphs we refer the reader to [1] and [2], respectively. For a complete study on domination we refer to [11] and [12], and for a general survey on kernels of digraphs we refer to [3] and [7].

Given a hypergraph $H = (E_1, \ldots, E_m)$, $V(H)$ is the *vertex set* of H. If $T \subseteq V(H)$ is such that $T \cap E_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$, we say that T is a *transversal* of H. Given a digraph D, $V(D)$ and $A(D)$ denote the *vertex set* and the *arc set* of D, respectively. The *inverse digraph* D^{-1} is obtained from D by reversing the direction of each arc. For every $x \in V(D)$, the *out-neighborhood* of x is the set $N^+(x) = \{y \in V(D) \mid (x, y) \in A(D)\}$. The *out-degree* $\delta^+(x)$ and the *in-degree* $\delta^-(x)$ is defined as the number of arcs with x as their tail and head, respectively. If $K \subseteq V(D)$ is such that $(x, y) \notin A(D)$ and $(y, x) \notin A(D)$ for all pairs $\{x, y\} \subseteq K$, we say that K is *stable*; if every vertex $x \in V(D) \setminus K$ has an out-neighbour $y \in K$, we say that K is *absorbent* (equivalently, the complement $V(D) \setminus K$ is *dominating*); if K is both stable and absorbent, we say that it is a *kernel*. A digraph D is called *kernel-perfect* if, and only if, every induced subdigraph of D has a kernel. If $C = (x_0, x_1, \ldots, x_{n-1}, x_0)$ is a directed cycle contained in D, a *pseudodiagonal* of C is an arc $(x_i, x_j) \subseteq A(D) \setminus A(C)$ such that $\{i, j\} \subseteq \{0, \ldots, n-1\}$, and a *diagonal* of C is a pseudodiagonal (x_i, x_j) of C such that $(x_j, x_i) \notin A(C)$.

2. Square hypergraphs and kernels of digraphs

A hypergraph $H = (E_1, \ldots, E_m)$ is *square* if $|V(H)| = m$. A hypergraph H has a *system of distinct representatives (SDR)* if for every edge $E_i \in H$ there exists a vertex $x_i \in E_i$ such that $x_i \neq x_j$ whenever $i \neq j$. It is convenient to view an SDR as a bijective map $f : V(H) \to H$ such that $x \in f(x)$ for every $x \in V(H)$.

There is a one-to-one correspondence between digraphs and square hypergraphs with a (fixed) SDR: Given a digraph D on $V = V(D)$, consider the function $f_D : V \to 2^V$ defined as $f_D(x) = \{x\} \cup N^+(x)$, and let $H_D = (f_D(x) \mid x \in V)$. Then H_D is a square hypergraph with the SDR $f_D : V \to H_D$. Notice that some of the edges of H_D may be repeated; for example, if D is complete then every edge of H_D contains all the vertices in V. Conversely, given a square hypergraph H on V with an SDR $f : V \to H$, let $D_{H,f}$ be the digraph with $V(D_{H,f}) = V$ and $A(D_{H,f}) = \{(x, y) \mid x \in V, y \in f(x) \setminus \{x\}\}$.

Given a square hypergraph H on V with an SDR $f : V \to H$, a set $U \subseteq V$ is *f-stable* in H if $y \notin f(x)$ for every pair $\{x, y\} \subseteq U$.

Lemma 1. *Let D be a digraph on V, H_D its associated hypergraph with the SDR $f_D : V \to H_D$, and $U \subseteq V$. Then*

1. *$U \subseteq V$ is absorbent in D if, and only if, U is a transversal of H_D;*
2. *$U \subseteq V$ is stable in D if, and only if, U is f_D-stable in H_D;*
3. *U is a kernel of D if, and only if, U is an f_D-stable transversal of H_D.*

Proof. To prove (1), let $U \subseteq V$ be absorbent in D. Then for every $x \in V$ either $x \in U$ or there is a vertex $y \in U$ such that $(x, y) \in A(D)$. So for every $x \in V$ either $x \in U$ or there is a vertex $y \in U$ such that $\{x, y\} \subseteq E$ for some $E \in H_D$, which implies that U is a transversal of H_D. Conversely, if U is a transversal of H_D then $U \cap E \neq \emptyset$ for every $E \in H$. Then for every $x \in V$, either $x \in U$ or $f(x) \cap U \neq \emptyset$, so there exists a vertex $y \in U$ such that $(x, y) \in A(D)$. This implies that U is absorbent in D.

To prove (2), note that if U is not f_D-stable in H_D, then there exists a pair $\{x, y\} \subseteq U$ such that $y \in f(x)$. Then $(x, y) \in A(D)$ and U is not stable in D. Conversely, if U is not stable in D, then $(x, y) \in A(D)$ for some $\{x, y\} \subseteq U$. Then $y \in f(x)$ and U is not f_D-stable in H_D.

Finally, (3) follows by combining (1) and (2). □

A hypergraph H is 2-*colourable* if its vertices may be partitioned into two classes in such a way that every edge which is not a loop contains vertices of both classes. For more information about hypergraph colouring, the reader is referred to [2].

Observation 2. *Let D be a digraph and H_D its associated hypergraph with the SDR f_D. If D has a kernel, then H_D is 2-colourable.*

Proof. Let K be a kernel of D. K is a transversal of H_D by Lemma 1, so $K \cap E \neq \emptyset$ for every $E \in H_D$. Suppose $K \cap E = E$ for some $E \in H_D$. Since K is f_D-stable, $y \notin f(x)$ for every pair $\{x, y\} \subseteq E$, which implies $E = \{x\}$. Hence H_D is 2-colourable. □

So a necessary condition for D to have a kernel is that H_D is 2-colourable. It is not a sufficient condition, however, as shown by the example in Figure 1. It seems natural to ask whether some stronger form of 2-colourability of H_D is sufficient for the existence of a kernel of D. In the remainder of this section we show that this is indeed true when the hypergraph H_D is balanced and uniform. Before we can present our results, we need some further definitions.

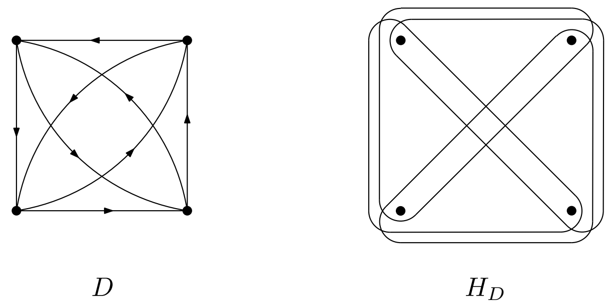

Figure 1: A digraph D without a kernel; the associated hypergraph H_D is 2-colourable.

Let D be a digraph and C a (not necessarily directed) cycle of D. An *obstruction* of C is a vertex $x \in V(C)$ such that $\delta^+_C(x) = 2$. The set of obstructions of C is denoted $\Omega(C)$. An Ω-*pseudodiagonal* of C is an arc $(u, v) \in A(D) \setminus A(C)$ such that $\delta^+_C(u) > 0$ and $\delta^-_C(v) > 0$; an Ω-*diagonal* of C is an Ω-pseudodiagonal (u, v) of C such that $(v, u) \notin A(C)$. An Ω-*symmetric arc* of C is an arc $(u, v) \in A(C)$ such that $u \notin \Omega(C)$ and $(v, u) \in A(D)$. The Ω-*length* of C is $\ell_\Omega(C) = |V(C)| - |\Omega(C)|$; a cycle is Ω-*odd* if $\ell_\Omega(C)$ is odd. A hypergraph H is *balanced* if every odd cycle in H has an edge containing at least three vertices of the cycle.

Proposition 3. *Let D be a digraph and H_D its associated hypergraph with the SDR f_D. The hypergraph H_D is balanced if, and only if, every Ω-odd cycle of D has at least one Ω-pseudodiagonal.*

Proof. Assume every Ω-odd cycle of D has at least one Ω-pseudodiagonal. Let $(x_1, E_1, \ldots, x_k, E_k, x_1)$ be an odd cycle of H_D, and take $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, where $y_i = f_D^{-1}(E_i)$. Consider the cycle C with $V(C) = X \cup Y$ and

$$\begin{aligned} A(C) &= \{(y_i, x_{i+1}), (y_i, x_i) \mid y_i \in Y \setminus X\} \cup \{(x_i, x_{i+1}) \mid x_i = y_i\} \\ &\quad \cup \{(x_{i+1}, x_i) \mid x_{i+1} = y_i\}. \end{aligned}$$

Notice that $\Omega(C) = Y \setminus X$ because no vertex may represent more than one edge, so $\ell_\Omega(C) = |X|$ and C must have an Ω-pseudodiagonal (u, v). Observe that $f_D(u) = E_i$ for some $i \in \{1, \ldots, k\}$, since (u, v) is an Ω-pseudodiagonal of D. So $f_D(u)$ contains at least three vertices of X, and since the cycle was chosen arbitrarily, it follows that H_D is balanced.

Conversely, assume that H_D is balanced and let C be an Ω-odd cycle in D. Then $X = V(C) \setminus \Omega(C)$ is the vertex set of an odd cycle $\Gamma = (x_1, E_1, \ldots, x_k, E_k, x_1)$ in H_D such that $E_i = f_D(y)$ for every subsequence (x_i, y, x_{i+1}) of C in which y is an obstruction. Since H_D is balanced, $|E_i \cap X| \geq 3$ for some $i \in \{1, \ldots, k\}$; let $x = f_D^{-1}(E_i)$. If $x \in X$ then $|N^+(x) \cap X| \geq 2$, and since only one of the arcs incident from x is in C, it follows that C has an Ω-pseudodiagonal. If $x \notin X$, then $|N^+(x) \cap X| \geq 3$, and since only two of the arcs incident from x are in C, it follows that C has an Ω-pseudodiagonal. □

A hypergraph H is *strongly unimodular* if it is balanced and has no odd cycles with one edge containing three vertices of the cycle and all other edges containing only two. The following analogue of Proposition 3 for strongly unimodular hypergraphs can be proved by a similar argument.

Proposition 4. *Let D be a digraph and H_D its associated hypergraph with the SDR f_D. The hypergraph H_D is strongly unimodular if, and only if, every Ω-odd cycle of D has at least two Ω-pseudodiagonals.*

Proof. Assume every Ω-odd cycle of D has two Ω-pseudodiagonals. Let $(x_1, E_1, \ldots, x_k, E_k, x_1)$ be an odd cycle of H_D, and define X, Y and C as in the proof of Proposition 3. Since C is Ω-odd, it has two Ω-pseudodiagonals (u_1, v_1) and (u_2, v_2). If $u_1 = u_2$, then $f_D(u_1)$ has four vertices of the cycle $(x_1, E_1, \ldots, x_k, E_k, x_1)$, as in the proof of Proposition 3. Similarly, if $u_1 \neq u_2$, both $f_D(u_1)$ and $f_D(u_2)$ have three vertices of the cycle $(x_1, E_1, \ldots x_k, E_k, x_1)$. Therefore, H_D is strongly unimodular.

Conversely, assume that H_D is strongly unimodular and let C be an Ω-odd cycle in D. Then $X = V(C) \setminus \Omega(C)$ is the vertex set of an odd cycle $\Gamma = (x_1, E_1, \ldots, x_k, E_k, x_1)$ in H_D such that $E_i = f_D(y)$ for every subsequence (x_i, y, x_{i+1}) of C in which y is an obstruction. Since H_D is strongly unimodular, one of the two following conditions holds:

(1) $|E_i \cap V(\Gamma)| \geq 3$ and $|E_j \cap V(\Gamma)| \geq 3$ for some $\{i, j\} \subseteq \{1, \ldots, k\}$;

(2) $|E_i \cap V(\Gamma)| \geq 4$ for some $i \in \{1, \ldots, k\}$.

If (1) holds, there is an Ω-pseudodiagonal incident from $f_D^{-1}(E_i)$ and an Ω-pseudodiagonal incident from $f_D^{-1}(E_j)$, as in the proof of Proposition 3. If (2) holds, then by the same argument there are two Ω-pseudodiagonals incident from $f_D^{-1}(E_i)$ (if $f_D^{-1}(E_i) \in X$, there is $x \in X$ such that $|N^+(x) \cap X| > 3$, and if $f_D^{-1}(E_i) \notin X$, there is $x \in \Omega(C)$ such that $|N^+(x) \cap X| > 4$). □

Balanced hypergraphs have many interesting and useful properties. In particular, Berge [2] proved the following result.

Theorem 5. *A balanced hypergraph has a good k-colouring for every $k \geq 2$: its vertices can be k-coloured so that the number of colours appearing in every edge E is exactly $\min\{|E|, k\}$.*

As a corollary, we can show that the digraph associated with a uniform balanced hypergraph contains many kernels.

Corollary 6. *Let D be a digraph with regular out-degree k such that every Ω-odd cycle has at least one Ω-pseudodiagonal. Then $V(D)$ can be partitioned into $k + 1$ kernels.*

Proof. The hypergraph H_D is $(k+1)$-uniform, and it is balanced by Proposition 3, so Theorem 5 implies that H_D has a good $(k+1)$-colouring. Since every colour class is a strongly stable transversal of H_D, it is an f-stable transversal of H_D for any SDR f. Therefore, Lemma 1 implies that each colour class is a kernel of D. □

3. A modification of Meyniel's conjecture

We now use the concept of obstructions to prove the main result of this paper, which may be viewed as a modification of Meyniel's original conjecture.

Theorem 7. *Let D be a digraph such that every 3-cycle is symmetric, every Ω-odd cycle of length four has at least two Ω-pseudodiagonals, and every Ω-odd cycle of length greater than four has at least two Ω-diagonals. Then D is a kernel-perfect digraph.*

The proof relies on several lemmas which we state and prove in this section. To make our arguments more concise, we introduce the following notation. Let D be a digraph and C an (undirected) cycle of D with $\{z, w\} \subseteq V(C)$. We denote by (z, C, w) the clockwise zw-path contained in C, and we write $P(C) = (i, j)$ whenever $\ell(C) \equiv i \pmod 2$ and $|\Omega(C)| \equiv j \pmod 2$. Let Γ and C be cycles of D such that $V(C) \subseteq V(\Gamma)$. We write $P_\Gamma(C) = (i, j)$ if $\ell(C) \equiv i \pmod 2$ and $|\Omega(\Gamma) \cap V(C)| \equiv j \pmod 2$. For $x \in V(C)$, the expression (x, Ω, C) means 'we may assume that x is an obstruction of C', while $(x, \neg\Omega, C)$ means 'we may assume that x is not an obstruction of C'.

A digraph D is defined to be *good* if every Ω-odd cycle $C \subseteq D$ such that $\ell(C) \geq 4$ satisfies at least one of the following properties:

(G1) There exists $x \in V(C)$ such that $\delta^+_{D[V(C)]}(x) \geq 3$;

(G2) There exists $x \in V(C)$ such that $\delta^-_{D[V(C)]}(x) \geq 3$;

(G3) C has a symmetric diagonal.

Lemma 8. *Let D be a digraph satisfying the following conditions:*

1. *Every directed 3-cycle is symmetric;*
2. *Every Ω-odd cycle of length at least four has at least two Ω-pseudodiagonals;*
3. *Every Ω-odd cycle of length at least five has at least two Ω-diagonals.*

Then D is good.

Proof. Let D be a digraph satisfying (1), (2) and (3), and let C be an Ω-odd cycle of D. We proceed by induction on the length of C. If $\ell(C) = 4$, $C = (x_0, x_1, x_2, x_3) \cup (x_0, x_3)$, where (x_0, x_1, x_2, x_3) is a directed path. Notice that no arc incident from x_0 or into x_3

is an Ω-pseudodiagonal, so the only possible choices are (x_0, x_2), (x_1, x_3), and (x_2, x_1). If $(x_0, x_2) \in A(D)$, $\delta^+_{D[V(C)]}(x_0) = 3$, and if $(x_1, x_3) \in A(D)$, $\delta^-_{D[V(C)]}(x_3) = 3$. So either (G1) or (G2) holds for every Ω-odd cycle of length 4.

Now suppose that every Ω-odd cycle C' of D such that $\ell(C') < m$ satisfies at least one of properties (G1), (G2) and (G3), and let C be an Ω-odd cycle of D such that $5 \leq \ell(C) = m$. By the hypothesis C has two Ω-diagonals $f_1 = (u, v)$ and $f_2 = (x, y)$. We may assume that neither f_1 nor f_2 is symmetric, and neither u nor x is an obstruction of C, for otherwise (G3) or (G1) would hold. Observe also that neither v nor y is an obstruction of C by the definition of Ω-pseudodiagonal.

We will consider several cases, according to the position of the Ω-diagonals f_1 and f_2. The analysis of a given case will finish when we find an Ω-odd cycle C' of D such that $\ell(C') < \ell(C)$ and $V(C') \subseteq V(C)$.

Case 1. f_1 and f_2 are parallel and have the same direction. Without loss of generality we may assume that $\{u, v\} \subseteq (y, C, x)$ and v precedes u in (y, C, x). We define the following cycles (see Figure 2): $C_1 = (x, C, y) \cup (x, y)$, $C_2 = (y, C, v) \cup (u, v) \cup (u, C, x) \cup (x, y)$, $C_3 = (v, C, u) \cup (u, v)$. Notice that since $\{u, v, x, y\} \cap \Omega(C) = \emptyset$, each obstruction of C is considered only once when we count the obstructions of C which are in the cycles C_1, C_2 and C_3. Observe also that x is an obstruction of exactly one of the cycles C_1 and C_2, and u is an obstruction of exactly one of the cycles C_2 and C_3.

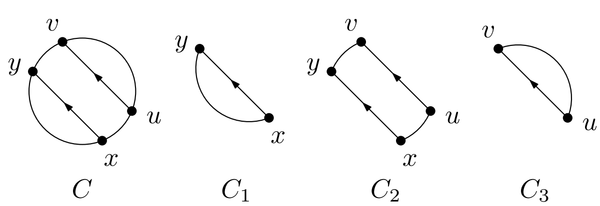

Figure 2: The cycles considered in Case 1.

We analyse several cases. Note that C_1 and C_3 play symmetric roles. Symmetric cases will only be considered once.

Case 1.1. $\ell(C)$ is even and $|\Omega(C)|$ is odd. We will analyse the cases for $P_C(C_1)$ and $P_C(C_2)$, assuming neither C_1 nor C_2 are Ω-odd.

1.1.1. $P_C(C_1) = (0, 0)$, $P_C(C_2) = (0, 0)$. Then $P_C(C_3) = (0, 1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (0, 1)$.

1.1.2. $P_C(C_1) = (0, 0)$, $P_C(C_2) = (0, 1)$. Then $P_C(C_3) = (0, 0)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (0, 1)$.

1.1.3. $P_C(C_1) = (0, 0)$, $P_C(C_2) = (1, 0)$. Then $P_C(C_3) = (1, 1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1, 0)$.

1.1.4. $P_C(C_1) = (0, 0)$, $P_C(C_2) = (1, 1)$. Then $P_C(C_3) = (1, 0)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1, 0)$.

1.1.5. $P_C(C_1) = (0,1)$, $P_C(C_2) = (0,1)$. Then $P_C(C_3) = (0,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (0,1)$.

1.1.6. $P_C(C_1) = (0,1)$, $P_C(C_2) = (1,0)$. Then $P_C(C_3) = (1,0)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1,0)$.

1.1.7. $P_C(C_1) = (0,1)$, $P_C(C_2) = (1,1)$. Then $P_C(C_3) = (1,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

1.1.8. $P_C(C_1) = (1,0)$, $P_C(C_2) = (0,0)$. Then $P_C(C_3) = (1,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

1.1.9. $P_C(C_1) = (1,0)$, $P_C(C_2) = (0,1)$. Then $P_C(C_3) = (1,0)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1,0)$.

1.1.10. $P_C(C_1) = (1,1)$, $P_C(C_2) = (0,1)$. Then $P_C(C_3) = (1,1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

Case 1.2. $\ell(C)$ is odd and $|\Omega(C)|$ is even. We will analyse the cases for $P_C(C_1)$ and $P_C(C_2)$.

1.2.1. $P_C(C_1) = (0,0)$, $P_C(C_2) = (0,0)$. Then $P_C(C_3) = (1,0)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1,0)$.

1.2.2. $P_C(C_1) = (0,0)$, $P_C(C_2) = (0,1)$. Then $P_C(C_3) = (1,1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

1.2.3. $P_C(C_1) = (0,0)$, $P_C(C_2) = (1,0)$. Then $P_C(C_3) = (0,0)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (0,1)$.

1.2.4. $P_C(C_1) = (0,0)$, $P_C(C_2) = (1,1)$. Then $P_C(C_3) = (0,1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (0,1)$.

1.2.5. $P_C(C_1) = (0,1)$, $P_C(C_2) = (0,0)$. Then $P_C(C_3) = (1,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

1.2.6. $P_C(C_1) = (0,1)$, $P_C(C_2) = (0,1)$. Then $P_C(C_3) = (1,0)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1,0)$.

1.2.7. $P_C(C_1) = (0,1)$, $P_C(C_2) = (1,0)$. Then $P_C(C_3) = (0,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (0,1)$.

1.2.8. $P_C(C_1) = (1,0)$, $P_C(C_2) = (1,0)$. Then $P_C(C_3) = (1,0)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying (u, Ω, C_2), $(u, \neg\Omega, C_3)$. Hence $P(C_3) = (1,0)$.

1.2.9. $P_C(C_1) = (1,0)$, $P_C(C_2) = (1,1)$. Then $P_C(C_3) = (1,1)$. So (x, Ω, C_1), $(x, \neg\Omega, C_2)$, implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

1.2.10. $P_C(C_1) = (1,1)$, $P_C(C_2) = (1,0)$. Then $P_C(C_3) = (1,1)$. So $(x, \neg\Omega, C_1)$, (x, Ω, C_2), implying $(u, \neg\Omega, C_2)$, (u, Ω, C_3). Hence $P(C_3) = (1,0)$.

Case 2. f_1 and f_2 are parallel and have opposite directions.

We may define the cycles $C_1 = (x, C, y) \cup (x, y)$, $C_2 = (y, C, u) \cup (u, v) \cup (v, C, x) \cup (x, y)$, and $C_3 = (u, C, v) \cup (u, v)$. Since each obstruction of C is considered only once when we count the obstructions of C which are in the cycles C_i for $i \in \{1,2,3\}$, x is an obstruction of exactly one of the cycles C_1 and C_2, and u is an obstruction of exactly one of the cycles C_2 and C_3. So we may proceed in exactly the same way as in Case 1.

Case 3. f_1 and f_2 cross, $u \in (y,C,x)$ and $v \in (x,C,y)$.

We define the following cycles (see Figure 3): $C_1 = (u,C,x) \cup (x,y) \cup (v,C,y) \cup (u,v)$, $C_2 = (x,C,v) \cup (u,v) \cup (y,C,u) \cup (x,y)$. Each obstruction of C is considered only once when we count the obstructions of C which are in the cycles C_i for $i \in \{1,2\}$. Moreover, both x and u are obstructions of exactly one of the cycles C_1 and C_2. Observe that C_1 and C_2 play symmetric roles. Symmetric cases will only be considered once.

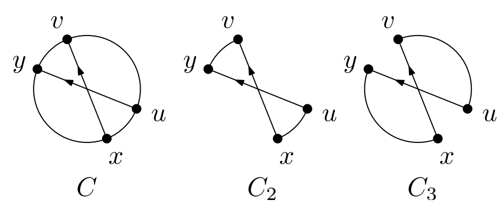

Figure 3: The cycles considered in Case 3.

Case 3.1. $\ell(C)$ is even and $|\Omega(C)|$ is odd. We will analyse the cases for $P_C(C_1)$, assuming C_1 is not Ω-odd.

3.1.1. $P_C(C_1) = (0,0)$. Then $P_C(C_2) = (0,1)$. Either none or both of x and u are obstructions of C_1, and the same applies to C_2, which implies $P(C_2) = (0,1)$.

3.1.2. $P_C(C_1) = (1,0)$. Then $P_C(C_2) = (1,1)$. Exactly one of x and u is an obstruction of C_1, and the same applies to C_2, which implies $P(C_2) = (1,0)$.

Case 3.2. $\ell(C)$ is odd and $|\Omega(C)|$ is even. We will analyse the cases for $P_C(C_1)$, assuming C_1 is not Ω-odd.

3.2.1. $P_C(C_1) = (0,0)$. Then $P_C(C_2) = (1,0)$. Either none or both of x and u are obstructions of C_1, and the same applies to C_2, which implies $P(C_2) = (1,0)$.

3.2.2. $P_C(C_1) = (0,1)$. Then $P_C(C_2) = (1,1)$. Exactly one of x and u is an obstruction of C_1, and the same applies to C_2, which implies $P(C_2) = (1,0)$.

Case 4. f_1 and f_2 cross, $u \in (x,C,y)$ and $v \in (y,C,x)$.

We define the following cycles: $C_1 = (v,C,x) \cup (x,y) \cup (u,C,y) \cup (u,v)$, $C_2 = (x,C,u) \cup (u,v) \cup (y,C,v) \cup (x,y)$. Since each obstruction of C is considered only once when we count the obstructions of C which are in the cycles C_1 and C_2, and both x and u are obstructions of exactly one of the cycles C_1 and C_2, the proof is identical to that of Case 3. This completes the proof of Lemma 8. □

Lemma 9. *Let D be a good digraph such that every directed 3-cycle is symmetric. Then every odd directed cycle of length at least five has a symmetric diagonal.*

Proof. Let D be a digraph satisfying the hypothesis of the theorem and let Γ be an odd directed cycle of length at least five. We proceed by induction on the length $\ell(\Gamma)$ of Γ.

Let $\Gamma = (x_0, x_1, x_2, x_3, x_4, x_0)$ be a 5-cycle and suppose Γ does not satisfy (G3). Then there is a vertex x_0 such that $\delta^+_{D[V(C)]}(x_0) \geq 3$ or $\delta^-_{D[V(C)]}(x_0) \geq 3$. Since the properties 'being good' and 'satisfying (G3)' hold for a digraph if, and only if, they hold for the inverse digraph, we may assume $\delta^+_{D[V(C)]}(x_0) \geq 3$. If $(x_0, x_3) \in A(D)$ then (x_0, x_3, x_4, x_0) is a directed 3-cycle, so (x_0, x_3) is a symmetric diagonal. Otherwise $(x_0, x_2) \in A(D)$ and $(x_0, x_4) \in A(D)$, and we have a 4-cycle $C = (x_0, x_2, x_3, x_4) \cup (x_0, x_4)$ with one obstruction. Since C is Ω-odd, it satisfies (G1), (G2) or (G3). In any case, C has a diagonal which is not (x_0, x_3): If $(x_3, x_0) \in A(D)$, then (x_0, x_2, x_3, x_0) is a directed 3-cycle, so (x_0, x_2) is a symmetric diagonal. If $(x_2, x_4) \in A(D)$, then (x_0, x_2, x_4, x_0) is a directed 3-cycle, so (x_2, x_4) is a symmetric diagonal. If $(x_4, x_2) \in A(D)$, then (x_4, x_2, x_3, x_4) is a directed 3-cycle, so (x_4, x_2) is a symmetric diagonal. Therefore Γ has a symmetric diagonal.

Now suppose that every directed cycle Γ' of odd length less than k has a symmetric diagonal, and let Γ be an odd cycle with $\ell(\Gamma) = k$. By hypothesis, Γ satisfies (G1), (G2) or (G3). Suppose Γ does not satisfy (G3), and let

$$\ell = \min\big\{\ell(x,\Gamma,y) \quad | \quad \{x,y\} \subseteq V(\Gamma) \text{ and } \exists u \in V(\Gamma) \setminus V(x,\Gamma,y) \text{ such that } \{(u,x),(u,y)\} \subseteq A(D) \text{ or } \{(x,u),(y,u)\} \subseteq A(D)\big\}$$

Observe that $\ell < \ell(\Gamma) - 2$. Let $\{z, w, v\}$ be a set of vertices for which the minimum is achieved, i.e., $\ell(z, \Gamma, w) = \ell$, $v \in V(\Gamma) \setminus V(z, \Gamma, w)$ and $\{(v, z), (v, w)\} \subseteq A(D)$ or $\{(z, v), (w, v)\} \subseteq A(D)$.

If there is a diagonal (x, y) of Γ such that $\ell(y, \Gamma, x)$ is even, then $(y, \Gamma, x) \cup (x, y)$ is a directed cycle of odd length less than k, which has a symmetric diagonal. Therefore we may assume that for every diagonal (x, y), the length $\ell(y, \Gamma, x)$ is odd.

Case 1: $\{(v, z), (v, w)\} \subseteq A(D)$. Since $\ell(v, \Gamma, z) \equiv \ell(v, \Gamma, w) \equiv 0 \pmod 2$, it follows that $C = (v, z) \cup (z, \Gamma, w) \cup (v, w)$ is an even cycle with exactly one obstruction, which must satisfy (G1), (G2) or (G3) because D is good. If C satisfies (G3), then Γ satisfies it too, so we may assume that C satisfies (G1) or (G2).

1.1: C satisfies (G1). Take $x \in V(C)$ such that $\delta^+_{D[V(C)]}(x) \geq 3$. Notice that $x \neq v$, for otherwise $\ell(z, \Gamma, w) > \ell$. If $(x, v) \in A(D)$ then $\ell(z, \Gamma, x)$ is even, otherwise $(v, z) \cup (z, \Gamma, x) \cup (x, v)$ is a directed cycle of odd length less than k. If its length is 3 it is symmetric, so (v, z) is a symmetric diagonal of Γ. If its length is larger, then by the induction hypothesis it has a symmetric diagonal, which is also a symmetric diagonal of Γ. Then $\ell(x, \Gamma, w)$ is even too, and $\ell(w, \Gamma, v)$ is odd since (v, w) is a diagonal of Γ. So $\ell(x, \Gamma, v)$ is odd and $(v, \Gamma, x) \cup (x, v)$ is a directed cycle of odd length less than k.

If $(x, v) \notin A(D)$, there exists a pair $\{x_0, x_1\} \subseteq V(z, \Gamma, w)$ such that $\{(x, x_0), (x, x_1)\} \subseteq A(D)$. If $\{x_0, x_1\} \subseteq V(z, \Gamma, x)$ or $\{x_0, x_1\} \subseteq V(x, \Gamma, y)$ then $\ell(z, \Gamma, w) > \ell$, so we may assume $x_0 \in V(z, \Gamma, x)$ and $x_1 \in V(x, \Gamma, y)$. As (v, z) and (x, x_1) are diagonals of Γ, it follows that $\ell(v, \Gamma, z) \equiv \ell(x, \Gamma, x_1) \equiv 0 \pmod 2$, so $\Gamma' = (v, z) \cup (z, \Gamma, x) \cup (x, x_1) \cup (x_1, \Gamma, v)$

is a directed cycle of odd length less than k, which implies that Γ has a symmetric diagonal.

1.2: C satisfies (G2). This case follows from 1.1 by considering the inverse digraph D^{-1}.

Case 2: $\{(z,v),(w,v)\} \subseteq A(D)$. This case follows directly from Case 1 by considering the inverse digraph D^{-1} instead of D. This completes the proof of Lemma 9 □

Let D be a digraph and C an odd directed cycle. The *set of poles* of C is defined as

$$p(C) = \{x \in V(C) \mid (w,x) \text{ is a pseudodiagonal of } C \text{ for some } w \in V(C)\}.$$

A vertex in $p(C)$ is a *pole* of C. A digraph D satisfies the *pole property* if, for every odd directed cycle $C = (x_1, \ldots, x_{2n+1}, x_1)$ of D, there are poles x_i, x_j, x_k, x_ℓ such that $i < j \leq k < \ell$ and $\ell(x_i, C, x_j) \equiv \ell(x_k, C, x_\ell) \equiv 1 \pmod 2$. Galeana-Sánchez and Neumann-Lara [10] proved the following result.

Theorem 10. *If a digraph satisfies the pole property then it is kernel-perfect.*

Lemma 11. *Let D be a digraph such that every 3-cycle is symmetric and every odd directed cycle of length at least five has a symmetric diagonal. Then D satisfies the pole property.*

Proof. We proceed by induction on the length of the odd cycle $C = (x_1, \ldots, x_{2n+1}, x_1)$. For $n = 1$ we have a 3-cycle in which every vertex is a pole, so the assertion holds.

Suppose that every odd directed cycle of length less than $2n+1$ has the desired poles, and let C be an odd directed cycle $C = (x_1, \ldots, x_{2n+1}, x_1)$ with $n > 1$. By hypothesis, C has a symmetric diagonal (x_r, x_s); we may assume $1 \leq r < s$ and $\ell(x_r, C, x_s) \equiv 1 \pmod 2$. Therefore $C'' = (x_s, C, x_r) \cup (x_r, x_s)$ is an odd directed cycle with $\ell(C'') < 2n+1$. By the induction hypothesis there are poles x_i, x_j, x_k, x_ℓ of C'' such that $i < j \leq k < \ell$ and $\ell(x_i, C'', x_j) \equiv \ell(x_k, C'', x_\ell) \equiv 1 \pmod 2$. Since $\ell(x_r, C, x_s) \equiv 1 \pmod 2$ it follows that $\ell(x_i, C, x_j) \equiv \ell(x_k, C, x_\ell) \equiv 1 \pmod 2$, and it is clear that x_i, x_j, x_k, x_ℓ are poles of C. This completes the proof of Lemma 11. □

An immediate corollary of Theorem 10 and Lemma 11 is the following.

Corollary 12. *Let D be a digraph such that every 3-cycle is symmetric and every odd directed cycle of length at least five has a symmetric diagonal. Then D is kernel-perfect.*

Theorem 7 now follows from Lemmas 8 and 9 and Corollary 12.

4. Concluding remarks

The correspondence between digraphs and square hypergraphs with an SDR helped us find a class of digraphs whose vertex set may be partitioned into kernels, as well as a new class of kernel-perfect digraphs. It is tempting to believe that the latter may be extended by relaxing some of the conditions. In particular, we would like to propose the following conjecture.

Conjecture 13. *Let D be a digraph. If every Ω-odd cycle of D has an Ω-pseudodiagonal, then D is kernel-perfect.*

Conjecture 13 seems difficult. It may be easier to prove one of the following two weaker conjectures.

Conjecture 14. *Let D be a digraph. If every Ω-odd cycle of D has an Ω-diagonal, then D is kernel-perfect.*

Conjecture 15. *Let D be a digraph. If every Ω-odd cycle of D has two Ω-pseudodiagonals, then D is kernel-perfect.*

Finally, we state an even weaker conjecture, which may be regarded as a starting point for dealing with the others.

Conjecture 16. *Let D be a digraph. If every Ω-odd cycle of D has two Ω-diagonals, then D is kernel-perfect.*

The tools developed in this paper allow us to view digraphs from the perspective of hypergraphs, and vice versa. Although this paper focuses on kernels, it might be interesting to apply these ideas to other aspects of digraphs and hypergraphs.

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