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Secondary and Internal Distances of Sets in Graphs II

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Abstract

For any given type of a set of vertices in a connected graph $G = (V, E)$, we seek to determine the smallest integers $(x, y : z)$ such that for all minimal (or maximal) sets S of the given type, where $|V| > |S| \geq 2$, every vertex $v \in V - S$ is within shortest distance at most x to a vertex $u \in S$ (called *dominating distance*), and within distance at most y to a second vertex $w \in S$ (called *secondary distance*). We also seek to determine the smallest integer z such that every vertex $u \in S$ is within distance at most z to a closest neighbor $w \in S$ (called *internal distance*). In this paper, a sequel to two previous papers [21, 18], we determine the secondary and internal distances $(2, y : z)$ for 16 types of sets, all of which are distance-2 dominating sets, that is, whose dominating distances are at most 2.

Keywords: distance-2 domination, 2-packings, irredundant sets, induced matching, acyclic matching, disconnected matching.

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1. Introduction

Let $G = (V, E)$ be a connected graph. The *open neighborhood of a vertex* $v \in V$ is the set $N(v) = \{u | uv \in E\}$ of vertices u adjacent to v in G, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is defined as $deg(v) = |N(v)|$. The *open neighborhood of a set* $S \subset V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of* S is the set $N[S] = \bigcup_{v \in S} N[v]$.

The *subgraph of a graph* $G = (V, E)$ *induced by a set* $S \subseteq V$ is the subgraph $G[S] = (S, E \cap (S \times S))$, whose vertex set is S, and whose edge set consists of all edges in E joining two vertices in S.

Definition 1. *A set S of vertices is a dominating set if $N[S] = V$, that is every vertex $v \in V$ is either in S, or is in $V - S$ and is adjacent to at least one vertex $u \in S$.*

The *distance* $d(u, v)$ between two vertices in a connected graph G equals the minimum number of edges in a path from vertex u to vertex v.

We will say that the *distance* $d(u, S)$ between a vertex $u \in V$ and a set $S \subset V$ equals the minimum value of $d(u, v)$ over all vertices $v \in S$. It follows, therefore, that for every vertex $u \in S$, $d(u, S) = 0$.

We define the *dominating distance* $dd(S)$ *of a set* S to equal $dd(S) = max\{d(v, S)|v \in V - S\}$, and if $dd(S) = k$, we say that S is a *distance-k dominating set* as every vertex $v \in V - S$ is distance at most k to at least one vertex in S.

Let $S = \{u_1, u_2, \ldots, u_k\}$ be any set of vertices in a graph G and let $v \in V - S$ be any vertex in $V - S$. Let us assume that the vertices in S have been ordered so that $d(v, u_1) \leq d(v, u_2) \leq \ldots \leq d(v, u_k)$. It follows that $d(v, S) = d(v, u_1)$. We define the *secondary distance* $sd(v, S)$ of a vertex $v \in V - S$ to S to equal $d(v, u_2)$, that is, $sd(v, S) = d(v, u_2)$. Similarly, we define the *secondary distance* $sd(S)$ of a set S to equal the maximum value of $sd(v, S)$ over all vertices $v \in V - S$.

Thus, if $dd(S) = x$ and $sd(S) = y$ then every vertex in $V - S$ is at most distance x to a closest vertex in S and at most distance y to a second closest vertex in S.

We will also be interested in determining the smallest value z such that every vertex $u \in S$ is at most distance z to another other vertex in S. We call this the *internal distance* $id(S)$ of S.

Therefore, we will say that a set S is an $(x, y : z)$ set if its dominating distance is at most x ($dd(S) \leq x$), its secondary distance is at most y ($sd(S) = y$), and its internal distance is at most z ($id(S) = z$). Notice that if a set S is an $(x, y : z)$ set, then it is also an $(x', y' : z')$ set for any $x' > x$, $y' > y$, or $z' > z$.

Using this terminology, it follows that a set S is a dominating set if and only if its dominating distance equals one, $dd(S) = 1$.

This paper is a sequel to one by Hedetniemi, Hedetniemi, Rall and Knisely [21] that introduced the concept of secondary domination in graphs, and a second paper by the authors of this paper [18] that determined the (1,y:z) distances of 31 types of dominating sets, that are summarized in the table below.

Type of Dominating Set	$(dd, sd : id)$
1. dominating	$(1, 4 : 3)$
2. independent dominating	$(1, 4 : 3)$
3. k-dependent dominating	$(1, 4 : 3)$
4. acyclic dominating	$(1, 4 : 3)$
5. bipartite dominating	$(1, 4 : 3)$
6. odd and externally odd dominating	$(1, 4 : 3)$
7. internally strong dominating	$(1, 4 : 3)$
8. open irredundant dominating	$(1, 4 : 3)$
9. restrained dominating	$(1, 4 : 3)$
10. 1*-moveable dominating*	$(1, 4 : 3)$
11. 2*-maximal independent*	$(1, 3 : 3)$
12. mobile dominating	$(1, 3 : 3)$
13. secure dominating	$(1, 3 : 3)$
14. maximal enclaveless	$(1, 3 : 3)$
15. vertex cover	$(1, 3 : 2)$
16. weakly connected {independent} dominating	$(1, 3 : 2)$
17. global offensive alliance	$(1, 3 : 2)$
18. global defensive alliance	$(1, 2 : 3)$
19. maximal $k \geq 1$*-dependent set*	$(1, 2 : 2)$
20. P_3 *dominating*	$(1, 2 : 2)$
21. maximal internally strong	$(1, 2 : 2)$
22. global powerful alliance	$(1, 2 : 2)$
23. total dominating	$(1, 2 : 1)$
24. connected dominating	$(1, 2 : 1)$
25. paired dominating and maximal matching	$(1, 2 : 1)$
26. maximal total/isolate-free matching	$(1, 2 : 1)$
27. maximal uniquely restricted matching	$(1, 2 : 1)$
28. total vertex cover	$(1, 2 : 1)$
29. 2*-dominating*	$(1, 1 : 2)$
30. maximal acyclic	$(1, 1 : 2)$
31. maximal bipartite	$(1, 1 : 2)$

In this paper we determine the (2,y:z) distances for 16 types of distance-2 dominating sets.

2. (2,7:5) Sets

In this section we define 8 types of sets, each of which is a (2,7:5) set. Thus, for each type of set S, each vertex $v \in V - S$ is within distance 2 of at least one vertex in S and within distance at most 7 to a second vertex in S, while every vertex $u \in S$ is guaranteed to be within distance at most 5 to another vertex in S.

In what follows, whenever we refer to a set $S \subset V$ of some type, we will always assume that (i) S is nontrivial, that is $|V| > |S| \geq 2$, and (ii) all graphs G are connected.

2.1. Distance-1 dominating sets

The secondary distance in the following theorem was proved by Hedetniemi, Hedetniemi, Rall and Knisely in [21], while the internal distance was proved in [18]. We provide the same proof here and use it as a model for all subsequent proofs in this paper.

Theorem 2. *Every dominating set S in a connected graph $G = (V, E)$ is a (1,4:3)-set.*

Proof. Let $S \subset V$ be any dominating set in a connected graph $G = (V, E)$. We will first show that $id(S) \leq 3$, that is, for any vertex $u \in S$, there exists another vertex $w \in S$ such that $d(u, w) \leq 3$. Assume that there exists a vertex $u \in S$ for which $d(u, S - \{u\}) > 3$. Let u, x, y be the first three vertices on a shortest path from vertex u to a vertex $w \in S - \{u\}$, where $x, y \in V - S$. Notice that vertex u cannot be adjacent to vertex y, else this is not a shortest path from u to w. But since S is a dominating set, vertex y must be adjacent to at least one vertex, say $z \in S$. It follows therefore that $d(u, z) \leq 3$, contradicting our assumption that $d(u, S - \{u\}) > 3$. Therefore, $id(S) \leq 3$.

Since S is a dominating set, every vertex $v \in V - S$ must be adjacent to at least one vertex, say $u \in S$. But since $id(S) \leq 3$, it follows that vertex u must be within distance three to another vertex in S. Therefore, vertex $v \in V - S$ must be within distance four to a second vertex in S, and we have that $sd(S) \leq 4$. □

The (1,4:3) bounds can be achieved. The set $S = \{x, y\}$ in Figure 1 is a (1,4:3) set.

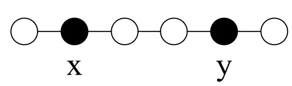

Figure 1: A (1,4:3) dominating set

Using Theorem 2 as a model, we can show that each of the following seven types of distance-2 dominating sets are (2,7:5) sets: distance-2 dominating sets, maximal 2-packings, maximal open 2-packings, perfect neighborhood sets, open perfect neighborhood sets, 1-maximal nearly perfect sets, and maximal efficient sets.

2.2. Distance-2 dominating sets

Theorem 3. *Every nontrivial distance-2 dominating set S in a graph G is a (2,7:5) set.*

Proof. Let $S \subset V$ be an arbitrary, nontrivial distance-2 dominating set in a graph $G = (V, E)$. By definition therefore, $dd(S) \leq 2$. We will show that $id(S) \leq 5$.

Assume that there exists a vertex $u \in S$ with $d(u, S - \{u\}) > 5$. Let $r \in S - \{u\}$ be a vertex in S nearest to u, that is $d(u, r) = d(u, S - \{u\})$. Let u, v, w, x be the first four vertices on a shortest path from u to r, where by assumption vertices $v, w, x \in V - S$. Consider vertex $x \in V - S$. Since S is a distance-2 dominating set, there must exist a vertex, say $s \in S$ with $d(x, s) \leq 2$. Since $s \neq u$, this means that $d(u, S - \{u\}) \leq 5$, contradicting our assumption that $d(u, S - \{u\}) > 5$.

It follows therefore that $id(S) \leq 5$ and that S is a (2,7:5) set, since every vertex $v \in V - S$ is within distance 2 of at least one vertex $u \in S$ ($dd(S) \leq 2$) and, in turn, u is within distance 5 of at least one other vertex $w \in S$, $w \neq u$. □

The set $S = \{x, y\}$ in the path P_{10} in Figure 2 is an exact (2,7:5) distance-2 dominating set, showing that these distance bounds can be achieved.

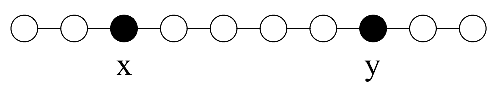

Figure 2: A (2,7:5) distance-2 dominating set

2.3. Maximal 2-packings and open 2-packings

Definition 4. *A set S of vertices is called a 2-packing if for every vertex $v \in V$, $|N[v] \cap S| \leq 1$, or equivalently, for every two vertices $u, w \in S$, $|N[u] \cap N[w]| = 0$.*

The term 2-packing comes from the observation that for any two vertices u, w in a 2-packing S, $d(u, w) > 2$.

Theorem 5. *Every nontrivial maximal 2-packing S in a graph G is a (2,7:5) set.*

Proof. Let S be a nontrivial maximal 2-packing in a graph G and assume that S is not a distance-2 dominating set. This means that there exists a vertex $x \in V - S$ such that $d(x, S) = 3$. Let u, v, w, x be the vertices on a shortest path between a vertex $u \in S$ and x. Because this is a shortest path from u to x, it follows that $N[x] \cap N[S] = \emptyset$. But this implies that the set $S \cup \{x\}$ is also a 2-packing, contradicting the maximality of S.

Since S is a distance-2 dominating set, it follows from Theorem 3 that S is a (2,7:5) set. □

The set $S = \{x, y\}$ in the path P_{10} in Figure 2 is a (2,7:5) maximal 2-packing.

Definition 6. *A set S of vertices is called an open 2-packing if for every vertex $v \in V$, $|N(v) \cap S| \leq 1$.*

In an open 2-packing S, every vertex $v \in V$ in the graph has at most one vertex in S in its open neighborhood $N(v)$.

Theorem 7. *Every nontrivial maximal open* 2*-packing S in a graph G is a (2,7:5) set.*

Proof. Let S be a nontrivial maximal open 2-packing in a graph G and assume that S is not a distance-2 dominating set. This means that there exists a vertex $x \in V - S$ such that $d(x, S) = 3$. Let u, v, w, x be the vertices on a shortest path between a vertex $u \in S$ and x. Because this is a shortest path from u to x, it follows that $N(x) \cap N(S) = \emptyset$. But this implies that the set $S \cup \{x\}$ is also an open 2-packing, contradicting the maximality of S.

Since S is a distance-2 dominating set, it follows from Theorem 3 that S is a (2,7:5) set. □

The set $S = \{x, y\}$ in the graph in Figure 3 is a (2,7:5) maximal open 2-packing.

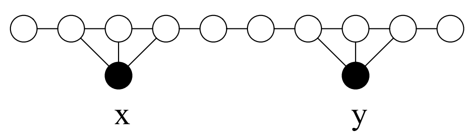

Figure 3: A (2,7:5) maximal open 2-packing

2.4. Perfect and open perfect neighborhood sets

Given a set S of vertices, we say that a vertex $v \in V$ is *perfect (with respect to S)* if $|N[v] \cap S| = 1$, that is, there is exactly one vertex in S in the closed neighborhood $N[v]$ of v.

Definition 8. *A set S is called a perfect neighborhood set if every vertex $v \in V$ is either perfect (with respect to S) or is adjacent to a perfect vertex.*

Perfect neighborhood sets were first defined and studied by Fricke, Haynes, Hedetniemi, Hedetniemi and Henning in [16] and have since been studied in [6, 8, 15, 19, 25].

Theorem 9. *Every nontrivial perfect neighborhood set S in a graph G is a (2,7:5) set.*

Proof. Let S be a nontrivial perfect neighborhood set. If a vertex $v \in V$ is perfect, then $d(v, S) \leq 1$, and if a vertex $w \in V$ is adjacent to a perfect vertex, then $d(w, S) \leq 2$. Therefore, since every vertex $v \in V$ satisfies $d(v, S) \leq 2$, S is a distance-2 dominating set, and it follows from Theorem 3 that S is a (2,7:5) set. □

The set $S = \{x, y\}$ in the path P_{10} in Figure 2 is a (2,7:5) perfect neighborhood set.

Given a set S of vertices, we say that a vertex $v \in V$ is *open perfect (with respect to S)* if $|N(v) \cap S| = 1$, that is, there is exactly one vertex in S in the open neighborhood $N(v)$ of v.

Definition 10. *A set S is called an open perfect neighborhood set if every vertex $v \in V$ is either open perfect (with respect to S) or is adjacent to an open perfect vertex.*

Theorem 11. *Every nontrivial open perfect neighborhood set S in a graph G is a (2,7:5) set.*

Proof. Let S be a nontrivial open perfect neighborhood set. If a vertex $v \in V$ is open perfect, then $d(v, S) \leq 1$, and if a vertex $w \in V$ is adjacent to an open perfect vertex, then $d(w, S) \leq 2$. Therefore, every vertex $v \in V$ satisfies $d(v, S) \leq 2$ and S is a (2,7:5) set. □

The set $S = \{x, y\}$ in the path P_{10} in Figure 2 is a (2,7:5) open perfect neighborhood set.

2.5. 1-maximal nearly perfect sets

We say that a set $S \subset V$ is *nearly perfect* if for every vertex $v \in V - S$, $|N(v) \cap S| \leq 1$. Notice that in this definition we require S to be a proper subset of V.

Definition 12. *A nearly perfect set S is called* 1*-maximal if for every vertex $v \in V - S$, the set $S \cup \{v\}$ is not a nearly perfect set.*

Nearly perfect sets were first defined and studied by Dunbar, Harris, Hedetniemi, Hedetniemi, McRae and Laskar in [13].

Theorem 13. *Every nontrivial* 1*-maximal nearly perfect set S in a graph G is a (2,7:5) set.*

Proof. Let $S \subset V$ be a 1-maximal nearly perfect set. Assume that S is not a distance-2 dominating set. Then there exists a vertex $v \in V - S$ with $d(v, S) \geq 3$. It follows therefore that $S \cup \{v\}$ is a nearly perfect set since no vertex $w \in V - S - \{v\}$ can be adjacent to both a vertex in S and vertex v. This contradicts the 1-maximality of S. It follows therefore that S is a distance-2 dominating set, and by Theorem 3 is a (2,7:5) set. □

The set $S = \{x, y\}$ in the graph in Figure 4 is a (2,7:5) 1-maximal nearly perfect set.

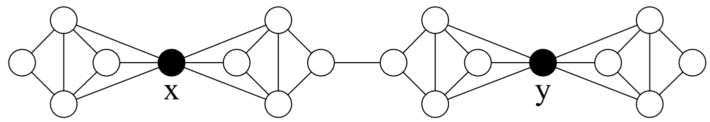

Figure 4: A (2,7:5) 1-maximal nearly perfect set

2.6. Maximal efficient sets

Let $S \subset V$ be a set of vertices in a graph $G = (V, E)$. We define $ED(S) = \{v|v \in V - S$ and $|N(v) \cap S| = 1\}$ to be the set of vertices in $V - S$ that are *efficiently dominated* by S. We then define the *efficiency of* S as $\epsilon(S) = |ED(S)|$, that is, the number of vertices in $V - S$ that are efficiently dominated by vertices in S, and the *efficiency of a graph* G equals $\epsilon(G) = max\{\epsilon(S)|S \subseteq V\}$.

Definition 14. *A set $S \subset V$ is a maximal efficient set if no proper superset of S has greater efficiency than S.*

The study of the efficiency of sets of vertices was initiated by Bernhard, Hedetniemi and Jacobs in [1], who presented a linear algorithm for computing the efficiency of an arbitrary tree and an NP-completeness result for the related efficiency question of a planar bipartite graph. The motivation for studying efficient sets originated from considerations of ethernet broadcasts, in which two (or more) vertices simultaneously *broadcast* a message to any vertex within range of their broadcasts. If a vertex is within range of two or more broadcasts, then we say that a *collision* occurs and the vertex does not receive either broadcast message. Thus, when considering efficiency, we seek to maximize the number of vertices that can simultaneously receive a broadcast message.

In this section we seek to determine the dominating, secondary and internal distances of a maximal efficient set S.

Theorem 15. *Every nontrivial, maximal efficient set is a (2,7:5) set.*

Proof. Let $S \subseteq V$ be a nontrivial maximal efficient set in a graph $G = (V, E)$. We must show that (i) $dd(S) \leq 2$, (ii) $sd(S) \leq 7$, and (iii) $id(S) \leq 5$.

(i) Assume that $dd(S) > 2$. Then there must exist a vertex $x \in V - S$ with $d(x, S) = 3$. Let $u \in S$ with $d(u, x) = 3$ and let u, v, w, x be the vertices on a shortest path between u and x, with $v, w, x \in V - S$. Because $d(x, S) = 3$, it follows that vertex w is not adjacent to any vertex in S (else $d(x, S) \leq 2$), and hence $\epsilon(S \cup \{x\}) > \epsilon(S)$, contradicting the maximality of S. Thus, $dd(S) \leq 2$ and S is a distance-2 dominating set.

From Theorem 3 we know that every distance-2 dominating set is a (2,7:5) set, and therefore, so is S. □

The set $S = \{x, y\}$ in the path P_{10} in Figure 2 is a (2,7:5) maximal efficient set.

3. (2,6:4) Sets

In this section we present the only example we know of a (2,6:4) set.

3.1. ve-dominating sets

In his 2007 PhD thesis, Lewis [23] introduced and studied the concept of *ve-domination* in graphs. In a graph $G = (V, E)$ we say that a vertex v *covers* every edge containing v, that is, every edge uv *incident* to v. We also say that v *ve-dominates* every edge incident to a vertex adjacent to v. Thus, if v, w, x are the vertices on a path from v to x, then we say that vertex v covers edge vw and ve-dominates edge wx.

Definition 16. *A set $S \subseteq V$ is a ve-dominating set in a graph $G = (V, E)$ if every edge in E is ve-dominated by some vertex in S.*

Theorem 17. *Every nontrivial ve-dominating set in a graph $G = (V, E)$ is a (2;6:4) set.*

Proof. Let S be a ve-dominating set in a graph $G = (V, E)$. We must show that (i) $dd(S) \leq 2$ and (ii) $id(S) \leq 4$. From this it will follow that $sd(S) \leq 6$.

(i) Assume that $dd(S) > 2$. This means that there exists a vertex $v \in V - S$ and a vertex $u \in S$ with $d(v, S) = 3 = d(u, v)$. Let u, w, x, v be the vertices on a shortest path from u to v, where $w, x \in V - S$. But this means that edge xv cannot be ve-dominated by any vertex in S, contradicting our assumption that S is a ve-dominating set. Therefore, $dd(S) \leq 2$.

(ii) Assume that $id(S) > 4$. Let $u \in S$ be a vertex for which $d(u, S - \{u\}) > 4$ and let $z \in S - \{u\}$ be a vertex for which $d(u, z) = d(u, S - \{u\})$. Let u, v, w, x, y be the first five vertices on a shortest path from u to z, where $v, w, x, y \in V - S$. We can infer that neither vertex w nor vertex x is adjacent to a vertex in $S - \{u\}$, else $d(u, S - \{u\}) \leq 4$, and neither w nor x is adjacent to vertex u, since this is a shortest path from u to z. From this it follows that edge wx cannot be ve-dominated by any vertex in S, contradicting our assumption that S is a ve-dominating set. Therefore, if S is a ve-dominating set, then $id(S) \leq 4$.

From (i) and (ii) it follows immediately that for any ve-dominating set S, $sd(S) \leq dd(S) + id(S) \leq 2 + 4 = 6$. □

The set $S = \{x, y\}$ in the path P_9 in Figure 5 is a (2,6:4) ve-dominating set.

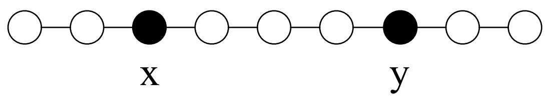

Figure 5: A (2,6:4) ve-dominating set

4. (2,5:5) Sets

In this section we present the only example we know of a (2,5:5) set. The proof of this result, and those that follow it, are considerably more involved than the proofs of the previous results.

4.1. Maximal open irredundant sets

Definition 18. *A set S is called open irredundant if every vertex $u \in S$ has at least one external private neighbor, that is, a vertex $w \in V - S$ for which $N(w) \cap S = \{u\}$.*

Theorem 19. *Every nontrivial, maximal open irredundant set is a (2,5:5) set.*

Proof. Let S be a nontrivial, maximal open irredundant set in a graph $G = (V, E)$. We must prove that: (i) $dd(S) \leq 2$, (ii) $sd(S) \leq 5$, and (iii) $id(S) \leq 5$.

(i) Assume that there exists a vertex $v \in V - S$ with $d(v, S) > 2$. Then there exists a vertex $z \in V - S$ with $d(z, S) = 3$. Let w, x, y, z be the vertices on a shortest path from S to z, where $w \in S$ and $x, y, z \in V - S$. Then $S \cup \{z\}$ is an open irredundant set, since y is an external private neighbor of z in $S \cup \{z\}$, and every vertex $u \in S$ having an external private neighbor in S, still has the same external private neighbor in $S \cup \{z\}$. This contradicts the assumption that S is a maximal open irredundant set. Therefore, $dd(S) \leq 2$.

(ii) Assume that $sd(S) > 5$, that is, there exists a vertex $v \in V - S$ and a vertex $u \in S$ such that $d(v, u) \leq 2$ (since $dd(S) \leq 2$), and $d(v, S - \{u\}) > 5$. Let $r \in S - \{u\}$ with $d(v, r) = d(v, S - \{u\})$, and let v, w, x, y, z be the first five vertices on a shortest path from v to r. There are five cases to consider, as shown in Figure 6.

Case 1. In this case we can conclude each of the following:

a. vertices u and s are not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$. Thus, t is an external private neighbor of u in S.

b. vertex y is not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 5$.

c. vertex u is not adjacent to vertex z and y is not adjacent to s, since this is a shortest path from v to r.

d. vertex z is not adjacent to any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 5$.

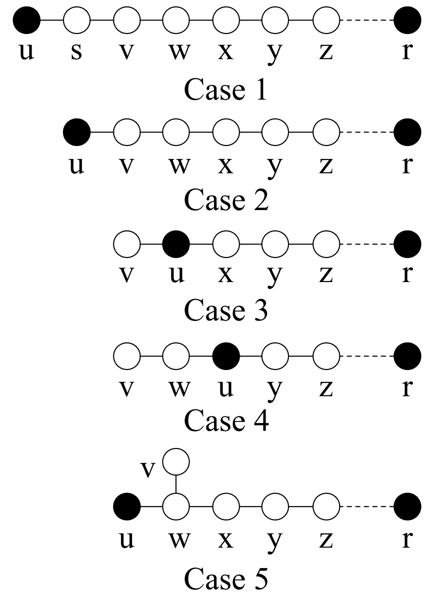

Figure 6: Five cases for $sd(S) \leq 5$

Thus, $S \cup \{y\}$ is an open irredundant set, since u has s as an external private neighbor and y has z as an external private neighbor. This contradicts the maximality of S.

Case 2. In this case we can conclude each of the following:

a. vertices u and v are not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 3$, and therefore v is an external private neighbor of u in S.

b. u is not adjacent to y since this is a shortest path from v to r.

c. vertices x and y are not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 5$.

d. x is not adjacent to v since this is a shortest path from v to r.

Therefore, v is an external private neighbor of u and y is an external private neighbor of x in $S \cup \{x\}$, and the set $S \cup \{x\}$ is an open irredundant set, contradicting the maximality of S.

Case 3. In this case we can conclude each of the following:

a. v is not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 2$, and therefore v is an external private neighbor of u in S.

b. u is not adjacent to y and x is not adjacent to v, since this is a shortest path from v to r.

c. x is not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$.

From this we can conclude that $S \cup \{x\}$ is an open irredundant set, since v is an external private neighbor of u and y is an external private neighbor of x. This contradicts the assumption that S is a maximal open irredundant set.

Case 4. In this case we can conclude each of the following:

a. v is not adjacent to u and w is not adjacent to y, since this is a shortest path from v to r.

b. y is not adjacent to any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$.

c. y is an external private neighbor of u in S.

d. neither v nor w are within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 3$.

From this it follows that $S \cup \{w\}$ is an open irredundant set, in which v is an external private neighbor of w and y is an external private neighbor of u. This contradicts the assumption that S is a maximal open irredundant set.

Case 5. In this case we can conclude each of the following:

a. none of u or v or w or x or y is within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 5$.

b. u is not adjacent to z and y is not adjacent to w, since this is a shortest path from v to r.

c. z is not adjacent to any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 5$.

It follows that the set $S \cup \{y\}$ is an open irredundant set, in which u has w as an external private neighbor and y has z as an external private neighbor, thereby contradicting the maximality of S.

From Cases 1 through 5 we can conclude that if S is a maximal open irredundant set, then $sd(S) \leq 5$.

(iii) Assume that $id(S) > 5$. Let $u \in S$ be a vertex for which $d(u, S - \{u\}) > 5$ and let $s \in S - \{u\}$ be a vertex for which $d(u, s) = d(u, S - \{u\}) > 5$. Let u, v, w, x, y, z be the first six vertices on a shortest path from u to s, where $v, w, x, y, z \in V - S$. In this case we can conclude each of the following:

a. none of u or v or w or x is within distance 2 of any vertex in $S - \{u\}$, else $d(u, S - \{u\}) \leq 5$, and therefore v is an external private neighbor of u in S.

b. u is not adjacent to y and x is not adjacent to v, since this is a shortest path from u to s.

c. vertex y is not adjacent to any vertex in $S - \{u\}$ else $d(u, S - \{u\}) \leq 5$.

It follows that $S \cup \{x\}$ is an open irredundant set, in which u has v as an external private neighbor and x has y as an external private neighbor. This contradicts the assumption that S is a maximal open irredundant set.

Thus, every maximal open irredundant set is a (2,5:5) set. □

The set $S = \{x, y\}$ in the graph in Figure 7 is a (2,5:5) maximal open irredundant set, showing that these distances are best possible for this class of sets.

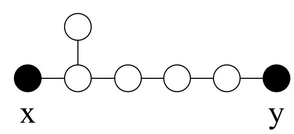

Figure 7: A (2,5:5) maximal open irredundant set

5. (2,5:3) Sets

In this section we present the only example we know of a (2,5:3) set.

5.1. 1-maximal restrained sets

A set $S \subseteq V$ in a graph $G = (V, E)$ is defined to be *restrained* if every vertex $v \in V - S$ is adjacent to at least one other vertex $w \neq v \in V - S$, that is, $N(v) \cap (V - S) \neq \emptyset$. A restrained set S is called 1-*maximal* if for every vertex $v \in V - S$ the set $S \cup \{v\}$ is not a restrained set.

Restrained sets that are also dominating sets were introduced by Domke, Hattingh, Hedetniemi, Laskar and Markus in [10] and later studied in [11, 12, 22, 26, 2, 9].

Theorem 20. *Every* 1*-maximal restrained set* S *in a graph* G *is a (2,5:3) set.*

Proof. Let S be a 1-maximal restrained set in a graph $G = (V, E)$. It will suffice to show that (i) $dd(S) \leq 2$ and (ii) $id(S) \leq 3$, since (i) and (ii) together imply that $sd(S) \leq 5$.

(i) $dd(S) \leq 2$. Assume that there exists a vertex $v \in V - S$ with $d(v, S) > 2$. Let $z \in V - S$ be a vertex at maximum distance from S, that is, $d(z, S) = max\{d(w, S) | w \in V - S\}$. It follows that $S \cup \{z\}$ must be a restrained set, since no vertex in $V - S$ can have z as its only neighbor in $V - S$, especially since $d(z, S) > 2$. This contradicts the 1-maximality of S. Thus, $dd(S) \leq 2$.

(ii) $id(S) \leq 3$. Assume that there exists a vertex $u \in S$ with $d(u, S - \{u\}) > 3$. Let $z \in S - \{u\}$ with $d(u, z) = d(u, S - \{u\})$, and let u, v, w, x be the first four vertices on a shortest path from u to z, where vertices $v, w, x \in V - S$. In this case, notice that vertex v is not within distance 2 of any vertex in $S - \{u\}$, else $d(u, S - \{u\}) \leq 3$. We claim that S is not a 1-maximal restrained set, because of the following three cases:

a. The only neighbors in G of vertex v are u and w. In this case it is easy to see that the set $S \cup \{v\}$ is a restrained set.

b. Vertex v is adjacent to a leaf $v' \in V - S$. In this case it is easy to see that the set $S \cup \{v'\}$ is a restrained set.

c. Vertex v is adjacent to one or more vertices in $V - S$, other than u and w, none of which are leaves. Because v is not within distance 2 of any vertex in $S - \{u\}$, it follows that each of these neighbors of v must in turn be adjacent to at least one other vertex in $V - S$. In this case, it follows that the set $S \cup \{v\}$ is a restrained set.

In either case above we contradict the assumption that S is a 1-maximal restrained set. Therefore, $id(S) \leq 3$.

Since $dd(S) \leq 2$ and $id(S) \leq 3$, it follows that $sd(S) \leq 2 + 3 = 5$. □

The set $S = \{x, y\}$ in the path P_6 in Figure 8 is a (2,5:3) 1-maximal restrained set and shows that these distances are best possible for this class of sets.

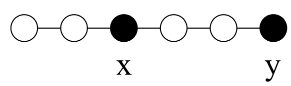

Figure 8: A (2,5:3) 1-maximal restrained set

6. (2,4:4) Sets

In this section we present three types of (2,4:4) sets.

6.1. Maximal matchable sets

Our first type of (2,4:4) sets are the maximal matchable sets, that were introduced and studied by Cockayne, Hedetniemi and Laskar in [7] in 1988.

Definition 21. *A set S is called matchable if there is a injection $M : S \to V - S$ such that for every vertex $u \in S$, u is adjacent to $M(u)$.*

Thus, every matchable set S naturally defines a matching M defined by the set of edges $u, M(u) \in E(G)$. Given a matching function M for a matchable set S, we say that a vertex $v \in V - S$ is *matched (by M)* if $v = M(u)$ for some vertex $u \in S$.

Theorem 22. *Every nontrivial maximal matchable set S in a graph G is a (2,4:4) set.*

Proof. Let $S \subset V$ be a nontrivial maximal matchable set with a matching function $M : S \to (V - S)$, and assume that S is not a distance-2 dominating set. This means that there exists a vertex $v \in (V - S)$ with $d(v, S) \geq 3$. Let w be any vertex adjacent to v on a shortest path from v to any vertex in S. It is easy to see that $S \cup \{v\}$ is a matchable set, since we can extend the matching function M on S to $S \cup \{v\}$ by defining $M(v) = w$. This contradicts the maximality of S. Therefore, S is a distance-2 dominating set and $dd(S) \leq 2$.

We next show that $sd(S) \leq 4$. Assume that $sd(S) > 4$, that is there exists vertices $u \in S$ and $v \in V - S$ with $d(v, u) \leq 2$, but $d(v, S - \{u\}) > 4$. Consider a shortest path from v to a second vertex $r \in S$, where $r \neq u$ and $d(v, r) = d(v, S - \{u\}) > 4$. Let v, w, x, y, z be the first five vertices on a shortest path from v to r. There are five cases to consider, as illustrated in Figure 9.

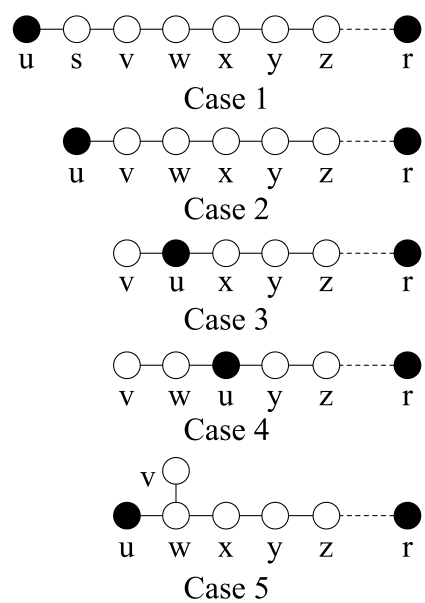

Figure 9: Five cases for $sd(S) \leq 4$

Case 1. $v, w, x, y, z \in V - S$ and vertex $u \in S$ is at distance 2 from v. In this case we can assume that v is not matched by the matching function M to a vertex in $S - \{u\}$, else $d(v, S - \{u\}) = 1$. We can also assume that vertices s and w are not matched by M to a vertex in $S - \{u\}$, else $d(v, S - \{u\}) = 2$. Consider next what vertex is matched to vertex u by M.

a. $M(u) = s$, in this case $S \cup \{w\}$ is a matchable set since vertex w can be matched to vertex v.

b. $M(u) = w$, in this case $S \cup \{s\}$ is a matchable set since vertex s can be matched to vertex v.

c. $M(u) \notin \{s, w\}$, in this case $S \cup \{s\}$ is a matchable set since vertex s can be matched to vertex v.

In either case above we contradict the assumption that S is a maximal matchable set.

Case 2. $v, w, x, y, z \in V - S$ and vertex $u \in S$ is adjacent to v. In this case, as in Case 1, neither w nor x nor y can be matched by M to a vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$, contradicting our assumption that $d(v, S - \{u\}) > 4$. Since this is a shortest path from v to r, we also know that vertex u is not adjacent to vertex y, and therefore M cannot match vertex u to vertex y. As in Case 1 above, consider next what vertex is matched to vertex u by M.

a. $M(u) = v$, in this case $S \cup \{w\}$ is a matchable set since vertex w can be matched to vertex x.

b. $M(u) = w$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y.

c. $M(u) \notin \{v, w\}$, in this case $S \cup \{v\}$ is a matchable set since vertex v can be matched to vertex w.

In either case above we contradict the assumption that S is a maximal matchable set.

Case 3. The first five vertices on a shortest path from v to r are v, u, x, y, z, where $u \in S$ and $v, x, y, z \in S$. In this case we can assume that v is not matched by M with a vertex other than possibly u, else $d(v, S - \{u\}) = 1$. We can also assume that neither x nor y is matched by M with a vertex other than u, else $d(v, S - \{u\}) \leq 4$, contradicting our assumption that $d(v, S - \{u\}) > 4$. Consider then what vertex is matched to vertex u by M.

a. $M(u) = v$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y.

b. $M(u) = x$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y while vertex u can be matched to vertex v.

c. $M(u) \notin \{v, x\}$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y and vertex u can be matched to vertex v.

In either case above we contradict the assumption that S is a maximal matchable set.

Case 4. The first five vertices on a shortest path from v to r are v, w, u, y, z, where $u \in S$ and $v, w, y, z \in S$. In this case we can assume that neither v nor w nor y are matched by M with a vertex other than u, else $d(v, S - \{u\}) \leq 4$. Consider then what vertex is matched to vertex u by M.

a. $M(u) = w$, in this case $S \cup \{w\}$ is a matchable set since vertex w can be matched to vertex v, while vertex u can be matched to vertex y.

b. $M(u) = y$, in this case $S \cup \{w\}$ is a matchable set since vertex w can be matched to vertex v.

c. $M(u) \notin \{w, y\}$, in this case $S \cup \{w\}$ is a matchable set since vertex w can be matched to vertex v.

In either case above we contradict the assumption that S is a maximal matchable set.

Case 5. A shortest path from v to $S - \{u\}$ passes in order through vertices v, w, x, y, z, where $v, w, x, y, z \in V - S$, w is adjacent to $u \in S$, while vertices v and u may or may not be adjacent. But in this case, neither v nor w nor x nor y can be matched by M to a vertex other than u, else $d(v, S - \{u\}) \leq 4$. Consider then what vertex is matched to vertex u by M.

a. $M(u) = w$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y.

b. $M(u) = x$ (u is adjacent to x), in this case $S \cup \{v\}$ is a matchable set since vertex v can be matched to vertex w.

c. $M(u) \notin \{w, x\}$, in this case $S \cup \{x\}$ is a matchable set since vertex x can be matched to vertex y.

In either case above we contradict the assumption that S is a maximal matchable set.

Thus, $sd(S) \leq 4$ and we only need to show that $id(S) \leq 4$. Assume that there exists a vertex $u \in S$ with $d(u, S - \{u\}) > 4$. Let $z \in S - \{u\}$ with $d(u, z) = d(u, S - \{u\})$, and let u, v, w, x, y be the first five vertices on a shortest path from u to z, where by assumption $v, w, x, y \in V - S$. We can assume that vertex u is not matched to either vertex w or x by the matching function M, since this is a shortest path from u to $S - \{u\}$. But in this case $S \cup \{w\}$ is a matchable set, since we can extend M to $M \cup \{w\}$, by defining $M(w) = x$, contradicting the maximal matchability of S.

Thus, every nontrivial, maximal matchable set S is a (2,4:4) set. □

The set $S = \{x, y\}$ in the graph in Figure 10 is a (2,4:4) maximal matchable set, showing that these distances are best possible for this class of sets.

6.2. Maximal irredundant sets

Given a set S of vertices, we can define three types of *private neighbors* with respect to S. We say that a vertex $u \in S$ is *its own, or self, private neighbor (or spn)* if u is not

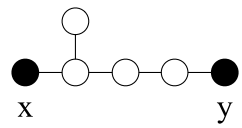

Figure 10: A (2,4:4) maximal matchable set

adjacent to any vertex in S, that is, $N(u) \cap (S - \{u\}) = \emptyset$. In this case we say that u is *independent in* S and is an isolated vertex in the induced subgraph $G[S]$.

As previously defined in Section 4.1, a vertex $u \in S$ has an *external private neighbor (or epn)* if it is adjacent to a vertex $v \in V - S$, and vertex v is not adjacent to any other vertex in S, that is, $N(v) \cap S = \{u\}$.

Finally, a vertex $u \in S$ has an *internal private neighbor (or ipn)* if it is adjacent to another vertex $w \in S$ and no other vertex in S is adjacent to w.

Definition 23. *A set S is called irredundant if every vertex $u \in S$ either is its own private neighbor or has at least one external private neighbor.*

Theorem 24. *Every nontrivial, maximal irredundant set is a (2,4:4) set.*

Proof. Let S be a nontrivial, maximal irredundant set in a graph $G = (V, E)$. We must prove that: (i) $dd(S) \leq 2$, (ii) $sd(S) \leq 4$, and (iii) $id(S) \leq 4$.

(i) Assume that there exists a vertex $v \in V - S$ with $d(v, S) > 2$. Then $S \cup \{v\}$ is an irredundant set, since v is independent in $S \cup \{v\}$ and is therefore its own private neighbor, and every vertex $u \in S$ having a private neighbor in S, still has the same private neighbor in $S \cup \{v\}$. This contradicts the assumption that S is a maximal irredundant set. Therefore, $dd(S) \leq 2$.

(ii) Assume that $sd(S) > 4$, that is, there exists a vertex $v \in V - S$ and a vertex $u \in S$ such that $d(v, u) \leq 2$ (since $dd(S) \leq 2$), and $d(v, S - \{u\}) > 4$. Let $r \in S - \{u\}$ with $d(v, r) = d(v, S - \{u\})$, and let v, w, x, y, z be the first five vertices on a shortest path from v to r.

As in the proof for maximal matchable sets, there are five cases to consider, as shown in Figure 9.

Case 1. We can conclude in this case that vertex v is not adjacent to vertex u, and that u is not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$. Thus, u is independent in S and is its own private neighbor (with respect to S).

We can also conclude that vertex v is not within distance 2 of any vertex in S other than u, else $d(v, S - \{u\}) \leq 2$. Thus, vertices u and v are independent in $S \cup \{v\}$ and $S \cup \{v\}$ is an irredundant set, since every vertex having a private neighbor in S has the same private neighbor in $S \cup \{v\}$. This contradicts the maximality of S.

Case 2. In this case we can conclude each of the following:

a. u is not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 3$, and therefore u is independent in S.

b. vertex x is not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

c. y is not adjacent to any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$, and y is not adjacent to u, since this is a shortest path from v to r, and therefore y is an external private neighbor of vertex x in the set $S \cup \{x\}$.

Therefore, the set $S \cup \{x\}$ is an irredundant set, since every vertex having a private neighbor in S has the same private neighbor in $S \cup \{x\}$, contradicting the maximality of S.

Case 3. In this case we can conclude each of the following:

a. v is not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 2$, and therefore v is an external private neighbor of u (with respect to S).

b. u is not adjacent to y, since this is a shortest path from v to r.

c. x is not within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

d. y is not adjacent to any vertex in S, else $d(v, S-\{u\}) \leq 4$.

From this we can conclude that $S \cup \{x\}$ is an irredundant set, since v is an external private neighbor of u and y is an external private neighbor of x, and every vertex having a private neighbor in S has the same private neighbor in $S \cup \{x\}$. This contradicts the assumption that S is a maximal irredundant set.

Case 4. In this case we can conclude each of the following:

a. v is not adjacent to u and w is not adjacent to y, since this is a shortest path from v to r.

b. y is not adjacent to any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

c. y is an external private neighbor of u in S.

d. neither v nor w are within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 3$.

From this it follows that $S \cup \{w\}$ is an irredundant set, in which v is an external private neighbor of w and y is an external private neighbor of u, and every vertex having a private neighbor in S has the same private neighbor in $S \cup \{w\}$. This contradicts the assumption that S is a maximal irredundant set.

Case 5. In this case we can conclude each of the following:

a. neither u nor v nor w nor x is within distance 2 of any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

b. u is independent in S and is therefore its own private neighbor in S.

c. y is not adjacent to any vertex in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

At this point we have to consider whether u is adjacent to v. If u is not adjacent to v then $S \cup \{v\}$ is an irredundant set in which both u and v are independent. If u is adjacent to v, then it cannot also be adjacent to y, since this is a shortest path from v to r. In this case $S \cup \{x\}$ is an irredundant set in which v is an external private neighbor of u and y is an external private neighbor of x.

Thus, if u is not adjacent to v, then $S \cup \{v\}$ is an irredundant set, and if u is adjacent to v then $S \cup \{x\}$ is an irredundant set. In either case we contradict the maximality of S.

From Cases 1 through 5 we can conclude that if S is a maximal irredundant set, then $sd(S) \leq 4$.

(iii) Assume that $id(S) > 4$. Let $u \in S$ be a vertex for which $d(u, S-\{u\}) > 4$ and let $r \in S-\{u\}$ be a vertex for which $d(u,r) = d(u, S-\{u\})$. Let u, v, w, x, y be the first five vertices on a shortest path from u to r, where $v, w, x, y \in V-S$. In this case we can conclude each of the following:

a. u is not within distance 2 of any vertex in $S-\{u\}$, else $d(u, S-\{u\}) \leq 2$, and therefore u is independent in S.

b. u is not adjacent to w, since this is a shortest path from u to r.

c. w is not within distance 2 of any vertex in $S-\{u\}$, else $d(u, S-\{u\}) \leq 4$.

It follows that $S \cup \{w\}$ is an irredundant set, in which both u and w are independent, and since every vertex having a private neighbor in S has the same private neighbor in $S \cup \{w\}$. This contradicts the assumption that S is a maximal irredundant set.

Thus, every maximal irredundant set is a (2,4:4) set. □

The set $S = \{w, x, y, z\}$ in the graph in Figure 11 is an exact (2,4:4) maximal irredundant set, showing that these distances are best possible for this class of sets.

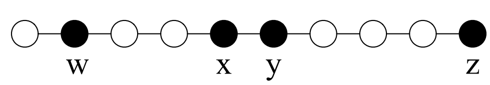

Figure 11: A (2,4:4) maximal irredundant set

6.3. External redundant sets

Given any set $S \subset V$ and any vertex $v \in S$, let $pn[v,S] = \{u | N[u] \cap S = \{v\}\}$ denote the set of private neighbors of vertex v with respect to set S. Given this notation, a set S is irredundant if and only if for every vertex $v \in S$, $pn[v,S] \neq \emptyset$.

Definition 25. *A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called external redundant if for every vertex $v \in V - S$, either*

(i) $pn[v, S \cup \{v\}] = \emptyset$*, or*

(ii) *there exists a vertex $w \in S$ with*

- **a.** $pn[w,S] \neq \emptyset$*, and*
- **b.** $pn[w, S \cup \{v\}] = \emptyset$.

In case (ii) above we say that the addition of v to S *annihilates* vertex w.

The concept of external redundant sets was introducted by McRae[24] in an attempt to characterize the conditions that make an irredundant set maximal. External redundant sets have since been studied in [3, 4, 5].

Theorem 26. *Every nontrivial, external redundant set is a (2,4:4) set.*

Proof. Let $S \subseteq V$ be an external redundant set in a graph $G = (V, E)$. We must show that (i) $dd(S) \leq 2$, (ii) $sd(S) \leq 4$, and (iii) $id(S) \leq 4$.

(i) $dd(S) \leq 2$. The definition of external redundant sets asserts that for every vertex $v \in V - S$, either (a) $pn[v, S \cup \{v\}] = \emptyset$, which implies that $d(v, S) = 1$, or (b) there exists a vertex $w \in S$ that has a private neighbor with respect to S, that is $pn[w, S] \neq \emptyset$, but does not have a private neighbor with respect to the set $S \cup \{v\}$, that is, $pn[w, S \cup \{v\}] = \emptyset$; this implies that $d(v, S) \leq 2$. Thus, $dd(S) \leq 2$.

(ii) $sd(S) \leq 4$. Assume that $sd(S) > 4$, that is, there exists a vertex $v \in V - S$ and a vertex $u \in S$ such that $d(v, u) \leq 2$ (since $dd(S) \leq 2$), and $d(v, S - \{u\}) > 4$. Let $r \in S - \{u\}$ with $d(v, r) = d(v, S - \{u\}) > 4$, and let v, w, x, y, z be the first five vertices on a shortest path from v to r, where $w, x, y \in V - S$.

As in the proof for maximal matchable sets and for maximal irredundant sets, there are again five cases to consider, as shown in Figure 9.

Case 1. Since we are assuming that $d(v, S - \{u\}) > 4$ and u is not adjacent to v, it follows that u is not adjacent to any vertex in $S - \{u\}$. Therefore, u is independent in S. It also follows that v is not within distance 2 of any vertex in $S - \{u\}$. Therefore, S is not external redundant, since $v \in pn[v, S \cup \{v\}] \neq \emptyset$ (v is its own private neighbor in $S \cup \{v\}$), and there cannot exist a vertex $w \in S$ with $pn[w, S] \neq \emptyset$ but $pn[w, S \cup \{v\}] = \emptyset$ (v cannot annihilate any vertex in S since such a vertex would have to be within distance 2 of v; no such vertex exists other than u and v does not annihilate u).

Case 2. In this case we can conclude each of the following:

a. u and v are not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 3$, and therefore u is independent in S.

b. x is not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$ and x is not adjacent to v since this is a shortest path from v to r.

c. y is not adjacent to any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$, and y is not adjacent to u, since this is a shortest path from v to r, and therefore y is an external private neighbor of vertex x in the set $S \cup \{x\}$.

It follows that S is not external redundant since (i) $y \in pn[x, S \cup \{x\}] \neq \emptyset$, and (ii) the only vertex in S within distance 2 of x is possibly u, but x does not annihilate u since u has v as an external private neighbor in $S \cup \{x\}$.

Case 3. In this case we can conclude each of the following:

a. v, u and x are not within distance 2 of any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$, and therefore v is an external private neighbor of u (with respect to S).

b. u is not adjacent to y and v is not adjacent to x, since this is a shortest path from v to r.

c. y is not adjacent to any vertex in S, else $d(v, S - \{u\}) \leq 4$.

From this we can conclude that S is not an external redundant set since (i) $y \in pn[x, S \cup \{x\}] \neq \emptyset$, and (ii) x does not annihilate any vertex in S (since it is not within distance 2 of any vertex in $S - \{u\}$ and since it does not annihilate u because u has v as an external private neighbor in $S \cup \{v\}$.

Case 4. In this case we can conclude each of the following:

a. v is not adjacent to u and w is not adjacent to y, since this is a shortest path from v to r.

b. Vertices v, w and y are not adjacent to any vertices in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$.

From this we can conclude that S is not an external redundant set since (i) $v \in pn[w, S \cup \{w\}] \neq \emptyset$, and (ii) vertex w cannot annihilate any vertex in S (it is not within distance 2 of any vertex in $S - \{u\}$ and it does not annihilate u since y is an external private neighbor of u in $S \cup \{w\}$).

Case 5. In this case we can conclude each of the following:

a. Either u is not adjacent to v or u is adjacent to v but is not adjacent to y, since this is a shortest path from v to r.

b. Vertices u, v, w and x are not adjacent to any vertices in $S-\{u\}$, else $d(v, S-\{u\}) \leq 4$.

c. y is not adjacent to any vertex in $S - \{u\}$, else $d(v, S - \{u\}) \leq 4$.

From this we can conclude that S is not an external redundant set since either (i) u is not adjacent to v, in which case $v \in pn[v, S \cup \{v\}] \neq \emptyset$, and vertex v cannot annihilate any vertex in $S - \{u\}$ and does not annihilate u, since u is independent in $S \cup \{v\}$, or (ii) u is adjacent to v, in which case $y \in pn[x, S \cup \{x\}] \neq \emptyset$ and vertex x cannot annihilate any vertex in $S - \{u\}$ and does not annihilate any vertex in $S - \{u\}$ and does not annihilate u since v is a private neighbor of u in $S \cup \{x\}$.

Thus in all cases 1 through 5 the assumption that $sd(S) > 4$ lead to a contradiction that S is an external redundant set. Therefore, if S is an external redundant set, then $sd(S) \leq 4$.

(iii) Assume that $id(S) > 4$. Let $u \in S$ be a vertex for which $d(u, S - \{u\}) > 4$ and let $z \in S - \{u\}$ be a vertex for which $d(u, z) = d(u, S - \{u\})$. Let u, v, w, x, y be the first five vertices on a shortest path from u to z, where $v, w, x, y \in V - S$. In this case we can conclude each of the following:

a. None of u, v and w is within distance 2 of any vertex in $S-\{u\}$, else $d(u, S-\{u\}) \leq 4$.

b. u is not adjacent to w or x or y, since this is a shortest path from u to z.

From this we can conclude that S is not an external redundant set since u and w are independent in $S \cup \{w\}$, and therefore $w \in pn[w, S \cup \{w\}] \neq \emptyset$, and w cannot annihilate any vertex in S, including u.

Thus, every external redundant set is a (2,4:4) set. □

The set $S = \{w, x, y, z\}$ in the graph in Figure 12 is a (2,4:4) external redundant set, showing that these distances are best possible for this class of sets.

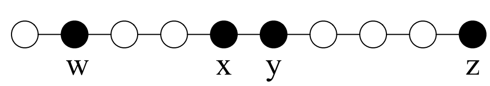

Figure 12: A (2,4:4) external redundant set

7. (2,3:1) and (2,2:1) Sets

In this section we present one example of (2,3:1) sets and two examples of (2,2:1) sets, each of which are sets of vertices saturated by a type of maximal matching.

A *matching* $M \subset E$ in a graph $G = (V, E)$ is a set of edges, no two of which have a vertex in common. Given a matching M and an edge $uv \in M$, we say that vertices u and

v are *saturated by* M. Therefore, let $V(M)$ denote the set of vertices saturated by M. In [18] it was shown (cf. Table 1) that the sets of vertices $V(M)$ saturated by maximal matchings, isolate-free matchings and uniquely restricted matchings are all (1,2:1) sets. In this section we show that the sets of vertices saturated by maximal induced matchings are (2,3:1) sets and the sets of vertices saturated by acyclic and disconnected matchings are (2,2:1) sets. The proofs of these three results are all simple and are based on the following, which we state without proof.

Lemma 27. *Let $V(M)$ be the set of vertices saturated by a matching M. Then if $V(M)$ is a distance-k dominating set, then $V(M)$ is a (k,k+1;1) set.*

Definition 28. *A matching M is called an induced matching if the induced subgraph $G[V(M)]$ consists of disjoint copies of K_2, or equivalently, contains no paths of length more than one.*

Definition 29. *A matching M is called an acyclic matching if the induced subgraph $G[V(M)]$ contains no cycles.*

Definition 30. *A matching M is called disconnected if the induced subgraph $G[V(M)]$ is disconnected.*

Matchings of these three types were studied by Goddard et al in [17].

Theorem 31. *The set $V(M)$ of vertices saturated by any maximal induced matching M is a (2,3:1) set.*

Proof. All that is necessary to prove this theorem is to show that the sets of vertices saturated by all maximal induced matchings are distance-2 dominating sets, and then appeal to Lemma 27. This is easily done. Assume that $S = V(M)$ is the set of vertices saturated by a maximal induced matching and assume that $dd(S) > 2$.

Then there exists a vertex $v \in V - S$ with $d(v, S) > 2$. In particular, there exists a vertex $x \in V - S$ with $d(x, S) = 3$. Let $u \in S$ with $d(u, x) = 3$, and let u, v, w, x be the vertices on a shortest path from u to x, where $v, w, x \in V - S$. It follows then that the set $S \cup \{w, x\}$ is the set of vertices saturated by an induced matching $M \cup \{wx\}$, contradicting the maximality of S. □

The graph P_4 in Figure 13 illustrates a (2,3:1) maximal induced matching, showing that these distances are best possible for this class of sets.

It is interesting to note, however, that the (2,3:1) bounds cannot be achieved by either maximal acyclic or maximal disconnected matchings.

Theorem 32. *The set $V(M)$ of vertices saturated by any maximal acyclic or maximal disconnected matching M is a (2,2:1) set.*

Proof. We must first prove that the sets of vertices saturated by all maximal acyclic matchings and all maximal disconnected matchings are distance-2 dominating sets. This is easily done. Assume that $S = V(M)$ is the set of vertices saturated by a maximal matching of either type (acyclic or disconnected) and assume that $dd(S) > 2$.

Then there exists a vertex $v \in V - S$ with $d(v, S) > 2$. In particular, there exists a vertex $x \in V - S$ with $d(x, S) = 3$. Let $u \in S$ with $d(u, x) = 3$, and let u, v, w, x be the vertices on a shortest path from u to x, where $v, w, x \in V - S$. It follows then that the set $S \cup \{w, x\}$ is the set of vertices saturated by a matching $M \cup \{wx\}$ of the same type (acyclic or disconnected), contradicting the maximality of S.

It only remains to show that the set of vertices $S = V(M)$ saturated by any maximal acyclic matching M (or any maximal disconnected matching) M satisfies $sd(S) \leq 2$.

Assume that $sd(S) > 2$. This means that there is a vertex $v \in V - S$ that is not adjacent to any vertex in S, but there is a vertex $u \in S$ such that $d(v, u) = 2$ and $d(v, S - \{u\}) > 2$. Let v, w, u be a path of length 2 from v to $u \in S$, where $w \in V - S$. Notice that vertex w cannot be adjacent to any vertex in S other than u, else $d(v, S - \{u\}) = 2$. Therefore, it follows that the matching obtained by adding the edge vw to M is still acyclic (or disconnected), since the degree of vertex w in the subgraph $G[S \cup \{v, w\}]$ is 2 and the degree of v in this subgraph is 1. This contradicts the assumption that M was a maximal acyclic (or a maximal disconnected) matching. Therefore, $sd(S) \leq 2$. □

The second and third graphs in Figure 13 illustrate a (2,2:1) maximal acyclic matching and a (2,2:1) maximal disconnected matching, showing that these distances are best possible for these classes of sets.

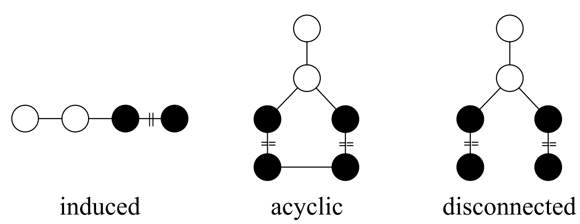

Figure 13: (2,3:1) maximal induced, and (2,2:1) maximal acyclic and maximal disconnected matchings

8. Summary

The following table summarizes the results in this paper.

Type of Set	$(dd, sd : id)$
1. distance-2 dominating set	$(2, 7 : 5)$
2. maximal 2-packing	$(2, 7 : 5)$
3. maximal open 2-packing	$(2, 7 : 5)$
4. perfect neighborhood set	$(2, 7 : 5)$
5. open perfect neighborhood set	$(2, 7 : 5)$
6. 1-maximal nearly perfect set	$(2, 7 : 5)$
7. maximal efficient set	$(2, 7 : 5)$
8. ve-dominating set	$(2, 6 : 4)$
9. maximal open irredundant set	$(2, 5 : 5)$
10. 1-maximal restrained set	$(2, 5 : 3)$
11. maximal matchable set	$(2, 4 : 4)$
12. maximal irredundant set	$(2, 4 : 4)$
13. external redundant set	$(2, 4 : 4)$
14. maximal induced matching	$(2, 3 : 1)$
15. maximal acyclic matching	$(2, 2 : 1)$
16. maximal disconnected matching	$(2, 2 : 1)$

The dominating, secondary and internal distances of several other types of irredundant sets will be determined in another paper by the authors. The proofs of these distances are more complex than those presented in this paper, since the number or the complexity of cases that have to be considered is greater.

8.1. Open-open irredundant sets

Definition 33. *A set S is called open-open irredundant if every vertex $u \in S$ has at least one external private neighbor or one internal private neighbor.*

8.2. Closed-open irredundant sets

Definition 34. *A set S is called closed-open irredundant if every vertex $u \in S$ has at least one private neighbor, either a self-private neighbor, an external private neighbor or an internal private neighbor.*

8.3. Total irredundant sets

Definition 35. *A set S is called total irredundant if every vertex $v \in V$ has at least one private neighbor, either a self-private neighbor or an external private neighbor, with respect to the set S.*

Total irredundant sets were first defined and studied by Favaron, Haynes, Hedetniemi, Henning, and Knisley in [14] and then studied algorithmically by Hedetniemi, Hedetniemi and Jacobs in [20].

8.4. R-annihilated sets

Given a set S of vertices in a graph $G = (V, E)$, we define the set $R = V - N[S]$ to equal the set of vertices not dominated by vertices in S. We say that a vertex $v \in R$ *annihilates* a vertex $u \in S$ if u has at least one external private neighbor with respect to S, but has no private neighbor with respect to the set $S \cup \{v\}$.

Definition 36. *A set S of vertices is called R-annihilated, or an ra-set, if every vertex in R annihilates at least one vertex in S.*

Notice that every maximal irredundant set S is an R-annihilated set, but not conversely.

R-annihilated sets and sets that are R-annililated and irredundant (*rai-sets*) have been studied by Cockayne, Favaron, Puech and Mynhardt [4] and Puech [25].

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