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Necessary Conditions for Strongly *-Graphs

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Abstract

We give new necessary conditions for a graph to be strongly *-graph. Second, we discuss the independence of these necessary conditions with known necessary conditions. Finally, we show that they are altogether not sufficient for a graph to be a strongly *-graph.

Keywords: Graph labeling, Strongly *-labeling and Strongly *-graph.

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1. Introduction

A variation of strong multiplicity of graphs is a strongly *-graph. A graph of order n is said to be a strongly *-graph if its vertices can be assigned the values $1, 2, ..., n$ in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j + ij$, all edges have different labels. Adiga and Somashekara [1] have shown that all trees, cycles, and grids are strongly *-graphs. They further consider the problem of determining the maximum number of edges in any strongly *-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs.

Seoud and Mahran [6] introduce some necessary conditions for a graph to be a strongly *-graph. They discuss the independence of these necessary conditions. Also, they show that they are altogether not sufficient for a graph to be a strongly *-graph.

Throughout this paper, we use the basic notations and conventions in graph theory as in [4], and in number theory as in [5], [3] and [7]. We use $| A |$ to denote the size of the set A, i.e., its number of elements, and $A - B$ to denote the usual difference between the sets A and B. All graphs here are simple, i.e., containing no loops or multiple edges.

2. Some necessary conditions

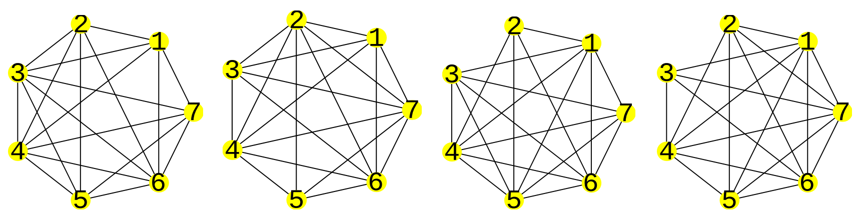
2.1. Properties of strongly *-graphs

Definition 2.1. [2] *A graph of order n is said to be a strongly *-graph if its vertices can be assigned the values $1, 2, ..., n$ in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j + ij$, all edges have different labels. A graph which is not strongly *-graph is said to be non-strongly *-graph.*

Definition 2.2. *A maximal strongly *-graph of n vertices is a strongly *-graph such that adding any new edge yields a non-strongly *-graph.*

Remark 2.3. *There exist many non-isomorphic maximal strongly *-graphs having the same number of vertices. If the number of maximal strongly *-graphs of order n is t, we denote them by $R^1(n), R^2(n), ..., R^t(n)$.*

Example 2.4. *The following graphs are the maximal strongly *-graphs of 7 vertices.*



Considering the four previous graphs without labeling, we see that they are all isomorphic except the first one.

Remark 2.5. *If for a labeling f, the edges i, j and k, m have the same label, then $(i + 1)(j + 1) = (k + 1)(m + 1)$.*

Theorem 2.6. [6]
Condition 1 *: If G is a graph of n vertices, which has number of edges more than $\lambda(n)$, then G is a non-strongly *-graph, where*

$$\lambda(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n} \sum_{k=1}^{m-1} \lfloor - \frac{\theta(m+1, k+1)}{\lfloor \sqrt{(m+1)(k+1) - 1} \rfloor - k} \rfloor \quad \text{and}$$

$$\theta(m, k) = \sum_{s=k+1}^{\lfloor \sqrt{mk-1} \rfloor} \lfloor \frac{\lfloor \frac{mk}{s} \rfloor}{\frac{mk}{s}} \rfloor$$

Condition 2 *: If the minimum degree of a graph G of n vertices is greater than the largest minimum degree in all corresponding maximal strongly *-graphs $\delta(n)$, then the graph is a non-strongly *-graph.*

Condition 3 *: If G is a graph of n vertices, which has number of vertices of degree $n-1$ more than $t(n)$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal strongly *-graphs of n vertices, then G is a non-strongly *-graph.*

Condition 4 *: If G is a graph of n vertices, and $K_{1+\chi(n)} \subseteq G$, where $\chi(n)$ is the order of the largest complete subgraph in all corresponding maximal strongly *- graphs, then G is a non-strongly *-graph.*

2.2. New necessary conditions

Definition 2.7. [7] *If n is a positive integer, the function $\tau(n)$ called the tau function, is defined to be the number of positive divisors of n, i.e., $\tau(n) := \mid \{d \in \mathbb{N} : d \mid n\} \mid$*

The tau function satisfies the following properties:

Theorem 2.8. [7] *If $(m, n) = 1$, then $\tau(mn) = \tau(m)\tau(n)$, i.e., τ is multiplicative.*

Theorem 2.9. [7] *If p is a prime number, then $\tau(p^\alpha) = \alpha + 1$.*

Corollary 2.10. [7] *If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, where the p's are distinct prime numbers, then $\tau(n) = \prod_{i=1}^{k}(\alpha_i + 1)$.*

Lemma 2.11. [3] *If n has no divisors less than or equal to $\sqrt{n}$, then n is a prime number.*

Definition 2.12. *For any two different positive integers $1 \le i, j \le n$, we define the set $V^n_{i*j} := \{(k, m) : i*j = k*m, \text{ and } 1 \le k < m \le n\}$, ($i*j$ means the product of $(i+1)(j+1)$).*

For example, if $n = 7$, $V^7_{1*5} = V^7_{2*3} = V^7_{12} = \{(1, 5), (2, 3)\}$, and $V^7_{2*7} = V^7_{3*5} = V^7_{24} = \{(2, 7), (3, 5)\}$.

The following lemma gives a formula for the order of the set V^n_{i*j}.

Lemma 2.13.

$$\mid V^n_{i*j} \mid = \begin{cases} \lfloor \frac{\tau(i*j)}{2} \rfloor - 1 & if\ n+1 \ge i*j \\ \\ \lfloor \frac{\tau(i*j)}{2} \rfloor - \sum_{d \mid i*j} \alpha(d) & if\ n+1 < i*j \end{cases}$$

where $\alpha(x) = \begin{cases} 1 & if\ x > n+1 \\ \\ 0 & if\ x \le n+1 \end{cases}$

Proof. Since the set $V^n_{i*j} := \{(k, m) : i*j = k*m, \text{ and } 1 \le k < m \le n\}$ and in the first case we have $n+1 \ge i*j$, it follows that $\mid V^n_{i*j} \mid$ is equal to the number of all distinct pairs of divisors of $i*j$ except the pair $(1, i*j)$ and the repeated pair $(\sqrt{i*j}, \sqrt{i*j})$ in case of $\sqrt{i*j}$ being an integer, also the number $\tau(i*j)$ is always an even number except in the case when $i*j = m^2, m \in \mathbb{N}$. So, $| V_{i*j}^n | = \left\lfloor \frac{\tau(i*j)}{2} \right\rfloor - 1$, if $n+1 \geq i*j$, hence the first case is proved. In the second case we have $n+1 < i*j$, it follows that $| V_{i*j}^n |$ is equal to the number of all pairs of the first case except those pairs in which there exists a divisor that exceeds $n+1$. So, $| V_{i*j}^n | = \left\lfloor \frac{\tau(i*j)}{2} \right\rfloor - \sum_{d|i*j} \alpha(d)$; if $n+1 < i*j$, hence the second case is proved. □

Definition 2.14. *Let $A_1, A_2, ..., A_m$ be sets satisfying that for each i, $| A_i | > 1$. We define the operation $X_{i=1}^m A_i := \{(\cup_{i=1}^m A_i) - \cup_{i=1}^m \{a_i\} : a_i \in A_i\}$.*

Remark 2.15. *After calculating all the sets V_{i*j}^n satisfying that $| V_{i*j}^n | > 1$, and calculating $X_{6 \leq i*j \leq n*(n-1), |V_{i*j}^n|>1} V_{i*j}^n$, we notice that, for each element $E \in X_{6 \leq i*j \leq n*(n-1), |V_{i*j}^n|>1} V_{i*j}^n$, we construct a maximal strongly *-graph by deleting all edges in the set E from the complete graph of n vertices.*

Now, we calculate the degree of the vertex labeled i in the maximal strongly *-graph associated with an element $E \in X_{6 \leq i*j \leq n*(n-1), |V_{i*j}^n|>1} V_{i*j}^n$ by the following lemma.

Lemma 2.16. *The degree of the vertex labeled i in the maximal strongly *-graph associated with an element $E \in X_{6 \leq i*j \leq n*(n-1), |V_{i*j}^n|>1} V_{i*j}^n$ is given by the following function $F(i, E) = n - 1 - \sum_{(r,s) \in E} \theta_i(r, s)$, where*

$$\theta_i(r, s) = \begin{cases} 1 & if\ r = i\ or\ s = i \\ \\ 0 & otherwise \end{cases}$$

Definition 2.17. *We define the following three sequences :*

1. *the sequence of the distinct degrees of vertices in a maximal strongly *-graph arranged in an ascending order, we call it the maximal degree sequence. In fact we have many different sequences of such type due to the existence of non-isomorphic maximal strongly *-graphs. We denote such a sequence by $D_{R^k(n)} = (d_i^k)$, where d_i^k is the ith degree of vertices in the k^{th} maximal strongly *-graph.*

2. *the sequences $C_{R^k(n)} = (c_i^k)$, where c_i^k is defined to be the number of vertices of degree at most d_i^k in the k^{th} maximal strongly *-graph, we call them the maximal strongly *-sequences.*

Definition 2.18. *For the given graph G, we define the graph sequences to be $B_G^k = (b_i^k)$, where b_i^k is the number of vertices of degree at most d_i^k in G.*

Example 2.19. *For $n = 7$, $V_{1*5}^7 = V_{2*3}^7 = V_{12}^7 = \{(1, 5), (2, 3)\}$, and $V_{2*7}^7 = V_{3*5}^7 = V_{24}^7 = \{(2, 7), (3, 5)\}$.*

So, $X_{6\leq i*j\leq 7*6,|V^7_{i*j}|>1}V^7_{i*j} = \{\{(1,5),(2,7)\},\{(1,5),(3,5)\},\{(2,3),(2,7)\},\{(2,3),(3,5)\}\}$.

The corresponding graphs are $R^1(7)$, $R^2(7)$, $R^3(7)$,$R^4(7)$ *are shown in Example 2.4, their distinct sequences are* $D_{R^1(7)} = \{5,6\}, C_{R^1(7)} = \{4,7\}$, $D_{R^2(7)} = D_{R^3(7)} = D_{R^4(7)} = \{4,5,6\}, C_{R^2(7)} = C_{R^3(7)} = C_{R^4(7)} = \{1,3,7\}$, *and hence* $t(7) = 4, \delta(7) = 5$.

Theorem 2.20. *(Condition 5) Let* G *be a simple graph for which there exists* i^k_0 *such that* $b^k_{i^k_0} < c^k_{i^k_0}$ *for every* k*, then* G *is a non-strongly *-graph.*

Proof. For a fixed k, suppose that there exists i^k_0 such that $b_{i^k_0} < c_{i^k_0}$, i.e., the number of vertices of degree at most $d_{i^k_0}$ in G is less than the number of vertices of degree at most $d_{i^k_0}$ in the corresponding k^{th} maximal strongly *-graph, which is equal to the number of the labels of those vertices in the corresponding k^{th} maximal strongly *-graph. Then, to distribute these labels on the vertices of G we must put them on vertices of degrees at most $d_{i^k_0}$. Hence there exists at least one label, say r^k_0, on a vertex of degree at most $d_{i^k_0}$ in the corresponding k^{th} maximal strongly *-graph, which must be given to a vertex of degree more than $d_{i^k_0}$ in G, say v^k_0. Then there exist three vertices w^k_0, u^k_0, z^k_0, where w^k_0 is adjacent with v^k_0, and has label, say m^k_0. Also, the two vertices u^k_0, z^k_0 are adjacent and having labels, say s^k_0 and t^k_0, satisfying $r^k_0 * m^k_0 = s^k_0 * t^k_0$. Hence the graph G is not a subgraph of the k^{th} maximal strongly *-graph. Since it happens for each k, it follows that G is not a subgraph of any maximal strongly *-graph. Hence G is a non-strongly *-graph. □

Now, we'll show that Condition 5 is stronger than conditions 1,2,3 in the sense that every non-strongly *-graph by these conditions is a non-strongly *-graph by Condition 5.

Corollary 2.21. *If* G *is a graph of* n *vertices and* m *edges such that,* $m > \lambda(n)$*, then for each* k*, there exists* i^k_0 *such that* $b_{i^k_0} < c_{i^k_0}$.

Proof. Suppose that the degree of a vertex v_i of G is $\rho(i)$, and suppose that m, the number of edges of G, is equal to $1 + \lambda(n)$, and by deleting an edge we get a strongly *-graph, i.e., G becomes a maximal strongly *-graph. Suppose that the edge which causes G to be a non-strongly *-graph with respect to the k^{th} maximal strongly *-graph is one of g_k, h_k, connecting the vertices v^k_r, w^k_s and y^k_t, z^k_u respectively, having the labels r, s, t, u, such that $r * s = t * u$. Without loss of generality, suppose that $r < s$. Then the degree of v^k_r after removing the edge g_k is $\rho(r) = d_{i^k_r}$ and before removing g_k the degree is $d_{i^k_r} + 1$ and the number of vertices of degree at most $d_{i^k_r}$ in G is less than that number in the corresponding k^{th} maximal strongly *-graph, i.e., there exists $i^k_0 = i^k_r$ such that $b_{i^k_0} < c_{i^k_0}$. □

Corollary 2.22. *If the minimum degree of the graph is greater than* $\delta(n)$*, which is defined in Condition 2, then for every* k*, there exists* i_0 *such that* $b_{i_0} < c_{i_0}$.

Proof. Since the minimum degree of the graph is greater than $\delta(n)$, the largest minimum degree in all corresponding maximal strongly *-graphs, then for every k, the number of vertices of degree at most d_1^k in the graph is equal to zero, which is less than the number of vertices of degree at most d_1^k in the corresponding k^{th} maximal strongly *-graph. It is clear that $i_0 = 1$, which satisfies that for every k, $0 = b_{i_0} < c_{i_0}$. □

Corollary 2.23. *If a graph G of n vertices has a number of vertices of degree $n-1$ more than $t(n)$ -which is defined in Condition 3- , then there exists i_0 such that $b_{i_0} < c_{i_0}$, for each k.*

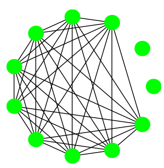
Proof. Suppose that the number of vertices of degree $n-1$ in G is greater than $t(n)$, which is defined in Condition 3, then the number of vertices of degree less than $n-1$ in G is $n - t(n)$, which is less than this number in all corresponding maximal strongly *-graphs. Then there exists i_0 (the prelast term) such that $b_{i_0} < c_{i_0}$, for every k. □

In the following table we give the distinct sequences of all maximal strongly *-graphs of n vertices.

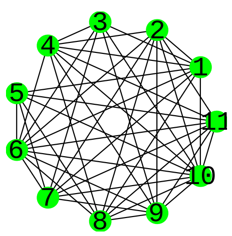
n	$1+\chi(n)$	$D_{R^i(n)}$	$C_{R^i(n)}$
5	5	{3,4}	{2,5}
6	6	{4,5}	{2,6}
7	7	{4,5,6} {5,6}	{1,3,7} {4,7}

Table 1

Example 2.24. *Condition 4 proves that the following graph is a non-strongly *-graph, while Condition 5 fails to decide that it is a non-strongly *-graph.*

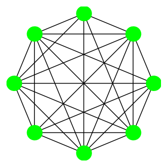


*($n = 11$, $G = K_9 \cup \overline{K_2}$) For Condition 5 : It can be shown that the following graph is a maximal strongly *-graph having the sequences $D_{R^k(11)} = \{6, 8, 10\}$, $C_{R^k(11)} = \{2, 9, 11\}$.*



Also, the corresponding graph sequence is $B_G^k = \{2, 11, 11\}$*. Hence there exists* k *satisfying that* $b_i^k \geq c_i^k$*, for every* i*. But for Condition 4 :* $K_{1+\chi(11)} = K_9 \subsetneqq G$*.*

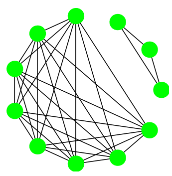
Example 2.25. *Condition 5 proves that the following graph is a non-strongly *-graph, while Condition 4 fails to decide that it is a non-strongly *-graph.*



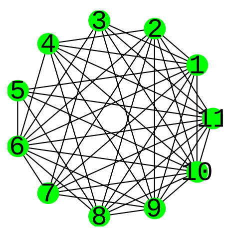
($n = 8$*) For Condition 4 :* $K_{1+\chi(8)} = K_8 \nsubseteq G$*. But for Condition 5 : It can be shown that the corresponding distinct sequences are* $D_{R^1(8)} = \{5, 7\}, C_{R^1(8)} = \{3, 8\}$*;* $D_{R^2(8)} = \{4, 6, 7\}, C_{R^2(8)} = \{1, 4, 8\}$*;* $D_{R^3(8)} = \{5, 6, 7\}, C_{R^3(8)} = \{2, 4, 8\}$*;* $D_{R^4(8)} = \{5, 6, 7\}, C_{R^4(8)} = \{1, 5, 8\}$*.*

Also, their corresponding graph sequences are $B_G^1 = \{1, 8\}$*,*$B_G^2 = \{0, 3, 8\}$*,*$B_G^3 = \{1, 3, 8\}$*,*$B_G^4 = \{1, 3, 8\}$*.*

Example 2.26. *Here we give an example of a non-strongly *-graph, but conditions 4 and 5 fail to decide that it is a non-strongly *-graph, i.e., they are altogether not sufficient for a graph to be a non-strongly *-graph.*



($n = 11$*,* $G = K_8 \cup K_3$*) For Condition 4 :* $K_{1+\chi(11)} = K_9 \nsubseteq G$*. For Condition 5 : It can be shown that the following graph is a maximal strongly *-graph having the sequences* $D_{R^k(11)} = \{5, 6, 7, 8, 9, 10\}$*,* $C_{R^k(11)} = \{1, 2, 4, 6, 9, 11\}$*.*



Also, the corresponding graph sequence is $B_G^k = \{3, 3, 11, 11, 11, 11\}$*. Hence there exists* k *satisfying that* $b_i^k \geq c_i^k$*, for every* i*. It remains to show that the graph* G *is a non-strongly *-graph.*

The edges which have the same labels are given by

$V^{11}_{1*5} = V^{11}_{2*3} = V^{11}_{12} = \{(1,5),(2,3)\}$; $V^{11}_{2*7} = V^{11}_{3*5} = V^{11}_{1*11} = V^{11}_{24} = \{(2,7),(3,5),(1,11)\}$; $V^{11}_{1*8} = V^{11}_{2*5} = V^{11}_{18} = \{(1,8),(2,5)\}$; $V^{11}_{1*9} = V^{11}_{3*4} = V^{11}_{20} = \{(1,9),(3,4)\}$; $V^{11}_{2*9} = V^{11}_{4*5} = V^{11}_{30} = \{(2,9),(4,5)\}$; $V^{11}_{3*9} = V^{11}_{4*7} = V^{11}_{40} = \{(3,9),(4,7)\}$; $V^{11}_{2*11} = V^{11}_{3*8} = V^{11}_{36} = \{(2,11),(3,8)\}$; $V^{11}_{3*11} = V^{11}_{5*7} = V^{11}_{48} = \{(3,11),(5,7)\}$; $V^{11}_{4*11} = V^{11}_{5*9} = V^{11}_{60} = \{(4,11),(5,9)\}$; $V^{11}_{5*11} = V^{11}_{7*8} = V^{11}_{72} = \{(5,11),(7,8)\}$.

We have ${}^{11}C_3 = 165$ *choices to label the graph by choosing three labels for the vertices of* K_3*. It can be shown that for each choice we have repetition in the edge labels. For example, by labeling the first two vertices of* K_3 *by 1,2 and the rest by 5,6,8,10 or 11, the edges (3,9) and (4,7) have the same label. Also, by labeling the first two vertices of* K_3 *by 1,5 and the rest by 6,7,8,9,10 or 11, the edges (1,5) and (2,3) have the same label. We delete the other cases because it makes the proof too lengthy. Hence* G *is a non-strongly *-graph.*

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