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Distance magic graphs of high regularity

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Abstract

A distance magic labeling of a graph G with n vertices is such a bijection f from the vertex set of G to the set of integers $\{1, 2, \dots, n\}$ that for every vertex in G the sum of labels of all adjacent vertices gives the same value k. A graph that allows such a labeling is a distance magic graph.

There is an elegant construction of r-regular distance magic graphs with an even number of vertices for all feasible values of r. For graphs of odd order certain necessary and certain sufficient conditions are known for the existence of a distance magic labeling. In this paper we show that an $(n-3)$-regular distance magic graph with n vertices exists iff $n \equiv 3 \pmod 6$ and that its structure is determined uniquely.

Moreover, we simplify the constructions from a recent paper by Fronček into a single construction and provide another sufficient condition for the existence a distance magic graph with an odd number of vertices.

Keywords: graph labeling, distance magic.

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1. Introduction and definitions

All graphs in this paper are finite, undirected without loops and multiple edges. Miller et al. [5] introduced the concept of distance magic labelings, formerly called 1-vertex magic vertex labelings. A *distance magic labeling* of a graph G with n vertices is a bijection $f : V(G) \to \{1, 2, \dots, n\}$ with the property that there exists an integer k such that for every vertex x is

$$w(x) = \sum_{y \in N(x)} f(y) = k,$$

where $N(x)$ is the set of all vertices adjacent to x. The constant k is the *magic constant* and $w(x)$ is the *weight* of x. If the magic constant k is evaluated for a certain graph G,

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we denote it by $k(G)$. A graph is *distance magic* if it admits a distance magic labeling. For convenience we identify vertices with their labels, thus e.g. vertex labeled 1 we call vertex 1 (see an example of a distance magic graph in Figure 1).

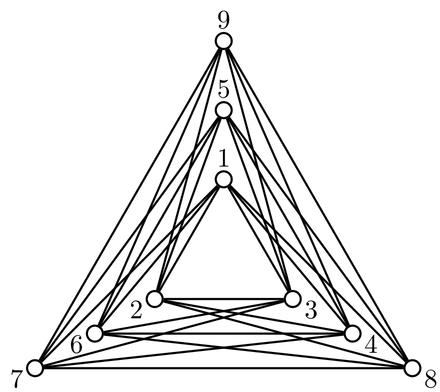

Figure 1: Distance magic labeling of a 6-regular graph with 9 vertices.

In this paper we focus only on regular graphs and we extend the results of Fronček [2] and Fronček, Kovářová and the first author [3, 4] who investigated the relation of regular distance magic graphs to scheduling of incomplete tournaments.

2. Known results

There is a recent survey on distance magic labelings by Arumugam, Fronček, and Kamatchi [1]. The majority of results on distance magic labelings concerns regular graphs. We summarize here only results relevant to this paper. The necessary conditions for a distance magic labeling of a given graph to exist are based mostly on counting arguments and parity. Sufficient conditions include usually constructions based on the partitioning of the set of consecutive integers (vertex labels) into classes with the same sum. In [5] it was shown that for a distance magic labeling f of G with n vertices

$$nk = \sum_{x \in V(G)} \deg(x) f(x), \tag{1}$$

where $\deg(x)$ is the degree of vertex x. The magic constant of regular graphs is determined uniquely by (1).

Proposition 2.1. *Let G be an r-regular distance magic graph with n vertices. The magic constant is $k(G) = r(n+1)/2$.*

The proof can be found e.g. in [5]. Recently, see [1], it was shown that the magic constant is determined uniquely in every distance magic graph. The following theorem gives a necessary and sufficient condition for a distance magic graph with an even number of vertices to exist. An analogous simple condition for regular distance magic graphs with an odd number of vertices is not expected to exist, see [2].

Proposition 2.2. [3] *Let n be even. An r-regular distance magic graph on n vertices exists if and only if $2 \le r \le n-2$, $r \equiv 0 \pmod 2$ and either $n \equiv 0 \pmod 4$ or $n \equiv r+2 \equiv 2 \pmod 4$.*

The composition $G[H]$ of graphs G and H arises from G by replacing every vertex of G by a copy of H, and replacing each edge of G by a complete bipartite graph $K_{|V(H)|,|V(H)|}$.

Proposition 2.3. [4] *Let H be an arbitrary r-regular graph with an odd number of vertices and let t be an odd positive integer. Then r is even and the graph $H[\overline{K_t}]$ is distance magic.*

In Section 3 we show that the only $(n-3)$-regular distance magic graph with n vertices is $K_{n/3}[\overline{K_3}]$.

The following proposition was proved by Fronček [2].

Proposition 2.4. [2] *Let $q, n > 1$ be odd, let $s \ge 1$ and $r = 2^s q$. Then an r-regular distance magic graph with m vertices exists whenever $r < \frac{2}{7}(m-2)$.*

The proof examines two cases and it is based on constructing a union of two $(2^s q)$-regular graphs G_x and G_y with disjoint vertex sets $V(G_x)$ and $V(G_y)$. Both G_x and G_y have to satisfy certain conditions to admit a distance magic labeling. In Section 4 we give a similar construction based on a more general approach which allows to find a distance magic labeling for many regular graphs with an odd number of vertices.

3. All $(n-3)$-regular distance magic graphs

In this section we describe all $(n-3)$-regular distance magic graphs by exploring their complement. Clearly, every distance magic graph G on n vertices is a factor of a complete graph K_n. Also, the complement of G (denoted by $\overline{G}$) is a factor of the complete graph K_n.

Theorem 3.1. *An $(n-3)$-regular distance magic graph G with n vertices exists if and only if $n \equiv 3 \pmod 6$. Further G is isomorphic to $K_{n/3}[\overline{K_3}]$.*

Proof. Notice, that by Proposition 2.2 there exists no $(n-3)$-regular distance magic graph with an even number of vertices, because $n-3$ is odd. Let n be odd and suppose G is an $(n-3)$-regular distance magic graphs with n vertices. The complement of G is a 2-regular graph $\overline{G}$.

Now, for convenience let G, $\overline{G}$, and K_n share the vertex set. By Proposition 2.1 the magic constant of G is $k = (n-3)(n+1)/2$. In K_n each vertex i is adjacent to all remaining vertices except itself, thus $w_{K_n}(i) = n(n+1)/2 - i$. Evaluating the weight of i in $\overline{G}$ we get

$$\begin{aligned} w_{\overline{G}}(i) &= w_{K_n}(i) - w_G(i) = w_{K_n}(i) - k \\ &= \frac{n(n+1)}{2} - i - \frac{(n-3)(n+1)}{2} = \frac{3(n+1)}{2} - i. \end{aligned}$$

There are no isolated vertices in $\overline{G}$. Take any two adjacent vertices $i, j \in V(\overline{G})$, then $w_{\overline{G}}(i) = 3(n+1)/2 - i$ and $w_{\overline{G}}(j) = 3(n+1)/2 - j$. The second vertex adjacent to i has the label $x = w_{\overline{G}}(i) - j = 3(n+1)/2 - i - j$ and the second vertex adjacent to j has the label $y = w_{\overline{G}}(j) - i = 3(n+1)/2 - j - i$. Obviously $x = y$, so any two adjacent vertices i and j in $\overline{G}$ have a common neighbor $x = 3(n+1)/2 - i - j$. Thus, $\overline{G}$ is formed by C_3 only, which implies that G is isomorphic to $K_{n/3}[\overline{K_3}]$ ($K_{3,3,\dots,3}$ with an odd number of partite sets due to Proposition 2.2).

It remains to show that $K_{n/3}[\overline{K_3}]$ is distance magic. This follows immediately from Proposition 2.3 by taking $H = K_{n/3}$ and $t = 3$. □

An example of an $(n-3)$-regular distance magic graph is in Figure 1.

4. Adding components

The following lemma shows that given a regular distance magic graph of certain odd order and certain regularity we can build an infinite class of larger distance magic graphs with the same regularity.

Lemma 4.1. *Let G be an r-regular distance magic graph with an odd number of vertices. Let $n = |V(G)|$. There exists an r-regular distance magic graph with m vertices for all odd $m = n + 2t$, if $2t \geq r + 2$ and if not both $r/2$ and t are odd.*

Proof. Let f be a distance magic labeling of G. By Proposition 2.1 the magic constant is $k(G) = r(n+1)/2$. Notice, that for odd order n the regularity r is even by parity principle.

Now we construct a graph G' with two components, we take G as one component. The other component is an r-regular graph G_o with $2t$ vertices. Let $t \geq 1 + r/2$. We take an arbitrary $(r/2)$-regular graph H with vertices $v_1, v_2, \dots, v_t$. Such graph always exists unless both t and $r/2$ are odd. Graph G_o will be the composition $H[\overline{K}_2]$. For each vertex v_i in H we denote by x_i, y_i the pair of vertices in the composition $H[\overline{K}_2]$ that corresponds to v_i. Let f' be the following labeling of G'.

$$f'(u) = \begin{cases} f(u) + t & u \in V(G), \\ i & u = x_i, \\ n + 2t + 1 - i & u = y_i. \end{cases}$$

We show that f' is a distance magic labeling of G'. Obviously, G' is an r-regular graph, because G is r regular and every vertex in $H[\overline{K}_2]$ is of degree $2\Delta(H) = 2r/2 = r$. Also, f' is a bijection $V(G) \to \{1, 2, \dots, n + 2t\}$ since the smallest t labels are assigned to vertices $x_1, x_2, \dots, x_t$, the highest t labels are assigned to $y_1, y_2, \dots, y_t$, and all n remaining labels are assigned to vertices in the component G.

It remains to show that every vertex u in G' has the same weight. For $u \in V(G)$, $w_{G'}(u) = \sum_{w \in N_{g'}(u)} f'(w) = \sum_{w \in N_G(u)} (f(w) + t) = k + rt = r(n+1)/2 + rt = r(n + 2t +$

$1)/2$. Because every vertex $u \in H[\overline{K}_2]$ is adjacent to $r/2$ pairs of vertices with the sum of labels $i+(n+2t+1-i) = n+2t+1$, we have for $u \in H[\overline{K}_2]$ $w_{G'}(u) = r/2 \cdot (n+2t+1) = r(n+2t+1)/2$. Thus, f' is a distance magic labeling of G'. □

Let n be odd and $r \equiv 0 \pmod 4$. If there exists an r-regular distance magic graph with n vertices, then it follows by Lemma 4.1 that there are finitely many odd orders m for which we cannot decide whether an r-regular graph with m vertices does exist. The highest possible odd order for which an r-regular distance magic graph does not exist is $n+r$. Recall the existence of regular distance magic graphs with an even number of vertices is settled by Proposition 2.2. Clearly, for orders at most $r+1$ no r-regular distance magic graphs exist, for a complete classification of orders it is enough to examine only $(n-3)/2$ different orders, namely $r+3, r+5, \ldots, n+r$ (we do not count order n among the unsolved cases). On the other hand, if $r \equiv 2 \pmod 4$, the existence of r-regular distance magic graphs with $m = n+4t+2$ vertices for $4t \geq r$ remains open, because the construction in Lemma 4.1 does not apply.

Proposition 2.4 can be obtained as a combination of Proposition 2.3 and Lemma 4.1. Let H_1 be K_{2^s+1} and let H_2 be any 2^s-regular graph with (2^s+3) vertices and let q be an odd positive integer. By Proposition 2.3 the graph $G_1 = H_1[\overline{K_q}]$ is an r-regular distance magic graph with $n_1 = (2^s+1)q$ vertices and $G_2 = H_2[\overline{K_q}]$ is an r-regular distance magic graph with $n_2 = (2^s+3)q$ vertices. Finally, by Lemma 4.1 there exists for each $i = 1, 2$ an r-regular distance magic graph for all orders $m = n_i + 2t_i$, if $2t_i \geq r$ and if not both $r/2$ and t_i are odd. Notice, that if t_1 is odd, then t_2 is even and vice versa, because n_1 and n_2 are not congruent modulo 4. Now following the same line of computations as in [2] we get $m = n_i + 2t_i \geq n_i + r + 2 = (2^s+1)q + 2^s q + 2 = (2^{s+1}+1)\frac{r}{2^s} + 2$ or $(2^{s+1}+3)\frac{r}{2^s} + 2$, respectively. We conclude

$$r \leq \frac{2^s}{2^{s+1}+3}(m-2) < \frac{2^s}{2^{s+1}+1}(m-2) \leq \frac{2}{7}(m-2).$$

The advantage of Lemma 4.1 to Proposition 2.4 is that it applies also for certain prime orders of the distance magic graph G. Moreover, the distance magic graph G' can also be of prime order. For example, the graph G in Figure 2 is a 4-regular distance magic graph with 17 vertices and by Lemma 4.1 it follows that for all odd orders higher or equal to 23, a 4-regular distance magic graph exists.

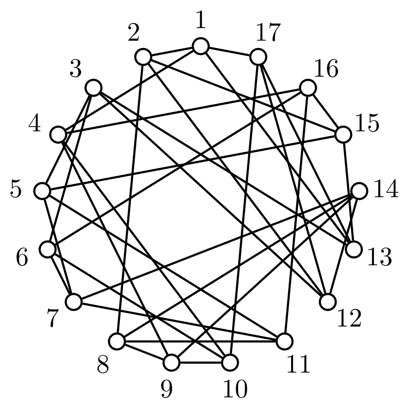

Figure 2: A 4-regular distance magic graph on 17 vertices with magic constant $k = 36$.

5. Conclusion

In the first part we studied the existence of dense regular distance magic graphs. Clearly the $(n-1)$-regular graph is the complete graph which is not distance magic. The existence of regular distance magic graphs of even order is settled completely by Proposition 2.2, see [3]. No $(n-2t)$-regular graph with an odd number of vertices n and arbitrary t exists by parity principle.

By Theorem 3.1 the structure of every $(n-3)$-regular distance magic graph is determined uniquely. It is isomorphic to the complete multipartite graph $K_{3,3,\ldots,3}$, where the number of partite set is odd. On the other hand notice, that the distance magic labeling of these graphs is not uniquely determined. In Figure 3 we have two different labelings of $\overline{K}_{3,3,3}$ and each gives a distance magic labeling of $K_{3,3,3}$ (the first example is the complement of the graph given in Figure 1).

Figure 3: Two examples of a labeled complement of 6-regular distance magic graph on 9 vertices.

The next natural step is to examine $(n-5)$-regular distance magic graphs. We do not expect the structure of such graphs be determined uniquely, because there exist several nonisomorphic 12-regular distance magic graphs with 17 vertices. Also a brute force search shows, that for n up to 13 no $(n-5)$-regular distance magic graph exists, while for $n = 15, 17, 19, 21$ such graphs exist.

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