AKCE Int. J. Graphs Comb., 9, No. 2 (2012), pp. 123-133

Some Interpretations of the Generalized Fibonacci Numbers

Andrzej Włoch
Rzeszów University of Technology
Faculty of Mathematics and Applied Physics
ul W.Pola 2, 35-359 Rzeszów, Poland.
e-mail: *awloch@prz.edu.pl*

Communicated by: S. Arumugam
Received 07 December 2011; accepted 14 May 2012

Abstract

In this paper we give some new interpretations of the generalized Fibonacci numbers and the generalized Lucas numbers. This interpretation is given with respect to the counting of special subfamilies of the set of n integers. Moreover we have applied these numbers for the graph interpretation of the number of all $\mathcal{H}$-matchings in special graphs, where $\mathcal{H}$ is a collection of some connected spanning subgraphs of K_k.

Keywords: Fibonacci numbers, lucas numbers, H-matching, counting.

2010 Mathematics Subject Classification: 05C69, 11B39.

1. Introduction

In general we use the standard terminology of the combinatorics and the graph theory.

The n-th Fibonacci number F_n is defined by $F_0 = 1$, $F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The n-th Lucas number L_n is defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.

In [4] Kwaśnik and Włoch introduced the concept of the generalized Fibonacci numbers $F(k,n)$ and the generalized Lucas numbers $L(k,n)$ in the following way:

Let $k \geq 2$ be an integer and let $X = \{1, ..., n\}$ be the set of n integers. Let $Y \subset X$ such that for each $i, j \in Y$ holds $|i - j| \geq k$. The generalized Fibonacci number $F(k,n)$ is the number of all subsets Y (including the empty set) and it has been proved that

$$F(k,n) = \sum_{p \geq 0} \binom{n - p - (p-1)(k-2) + 1}{p}.$$

The generalized Fibonacci numbers $F(k, n)$ were defined also by the following recurrence relation

$$F(k,n) = n+1 \text{ for } n = 0, 1, ..., k-1 \text{ and}$$
$$F(k,n) = F(k,n-1) + F(k,n-k), \text{ for } n \geq k. \tag{1}$$

Moreover the generalized Lucas numbers $L(k, n)$ were defined similarly in the following way:

Let $Y^* \subset X$ such that for each $i, j \in Y^*$ holds $k \leq |i-j| \leq n-k$. Then the generalized Lucas numbers $L(k, n)$ is the number of all subsets Y^* (including the empty set) and it has been proved that $L(k,n) = 1 + n + \sum\limits_{p \geq 2} \frac{n}{p}\binom{n-p(k-1)-1}{p-1}$.

Further the generalized Lucas numbers have the recurrence form

$$L(k,n) = n+1, \text{ for } n = 0, 1, ..., 2k-1 \text{ and}$$
$$L(k,n) = (k-1)F(k, n-(2k-1)) + F(k, n-(k-1)), \text{ for } n \geq 2k.$$

Recently some identities for the generalized Fibonacci numbers and the generalized Lucas numbers were given in [5] and [6]. In particular in [6] more comfortable recurrence relation for the generalized Lucas numbers were proved, too, namely, for $n \geq 2k$ holds, $L(k,n) = L(k,n-1) + L(k,n-k)$.

It is easy to see that $F(2,n) = F_n$, for $n \geq 0$ and $L(2,n) = L_n$, for $n \geq 3$.

In this paper we give new interpretations of the generalized Fibonacci numbers and the generalized Lucas numbers. These interpretations concern counting of special families of subsets of n integers. Next we give the graph interpretation of these families and the generalized Fibonacci numbers, too. The graph interpretation is connected with counting of H-matchings in special graphs.

Let G be a simple, undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. By a P_n and C_n we denote an n-vertex path and an n-vertex cycle, respectively.

Let G and H be two graphs. By an H-matching M of G we mean a subgraph of G such that all connected components of M are isomorphic to H. Note that the empty set is an H-matching, too, for every H. If M is also an induced subgraph of G, then the H-matching is called induced (see Fig.1 for example).

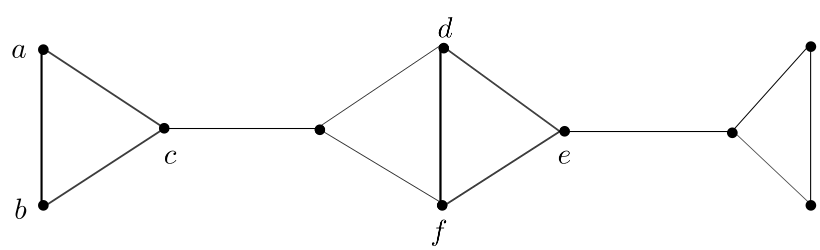

Fig. 1. An H-matching and an induced H-matching $\{\{a,b,c,\},\{d,e,f\}\}$ for $H = C_3$.

It is easy to see that if $H = K_2$, then K_2-matching is a matching in the classical sense. If $H = K_1$, then an induced K_1-matching is an independent set in the classical sense.

Firstly we count the number of special families of the set of n integers and we give the graph interpretation of the generalized Fibonacci numbers and the generalized Lucas numbers with respect to the number of P_k-matchings of graphs P_n and C_n, respectively. Next using these results and some classical operation of graph we give more general results and also we study the concept of $\mathcal{H}$-matchings which generalize H-matchings, defined as above.

For graph concepts not defined here see [1] and [2].

2. Generalized Fibonacci numbers and P_k-matchings

Let n, k be integers, $n \geq 2$, $k \geq 2$ and $X = \{1, 2, ..., n\}$. Let $\mathcal{X} = \{\{i, i+1, ..., i+k-1\}; i = 1, ..., n-k+1\}$ be the family of subsets of X. Let $\mathcal{Y} \subset \mathcal{X}$ be the subfamily of $\mathcal{X}$ such that

(i) $|\mathcal{Y}| = p$, for fixed $p \geq 0$ and

(ii) for each $Y_i, Y_j \in \mathcal{Y}$, $i \neq j$ holds $Y_i \cap Y_j = \emptyset$.

Let $g(k, n, p)$ denote the number of p-elements subfamilies $\mathcal{Y}$ and next let $G(k, n) = \sum_{p \geq 0} g(k, n, p)$.

Firstly we give a direct formula for the numbers $g(k, n, p)$ and $G(k, n)$, respectively.

Theorem 2.1. *Let n, k, p be integers, $n \geq 2$, $k \geq 2$, $p \geq 0$. Then $g(k, n, p) = \binom{n-(k-1)p}{p}$.*

Proof. Let $n \geq 2$, $k \geq 2$. Our aim is to calculate the number of all subfamilies $\mathcal{Y}$, having exactly p subsets and satisfying the condition (ii), considering also the empty subfamily, for $p = 0$. Suppose that $\mathcal{Y}$ is one of such subfamily of the family $\mathcal{X}$. For convenience, instead of the subfamily $\mathcal{Y}$ we consider the sequence $\alpha = (a_1, ..., a_n)$ such that

(a) $a_i = \begin{cases} 1 & \text{if } i \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$ and

(b) $\sum_{i=1}^{n} a_i = pk$.

Clearly there are p disjoint subsequences $\alpha_1, ..., \alpha_p$ of 1's such that $\alpha_i \cap \alpha_j = \emptyset$ for each $1 \leq i, j \leq p$. To calculate the number of all such sequences we replace each sequence of length k containing 1's by exactly one word 1. So, we study the subsequence $\beta = (b_1, ..., b_{n-(k-1)p})$. To build a sequence α first we put exactly p elements 1 on $n - (k-1)p$ possible places and from the fundamental combinatorial statements we can do it on $\binom{n-(k-1)p}{p}$ ways. Next each element 1 in the sequence β we replace by a subsequence $1, 1, ..., 1$ containing k elements 1. This replacing is unique, so there is exactly $\binom{n-(k-1)p}{p}$ sequences α. From the above considerations we obtain $g(k, n, p) = \binom{n-(k-1)p}{p}$. □

From this previous investigation it follows:

Corollary 2.2. *Let $n \geq 2$, $k \geq 1$ be integers. Then* $G(k,n) = \sum\limits_{p\geq 0} \binom{n-(k-1)p}{p}$.

The numbers $g(k,n,p)$ and $G(k,n)$ we can give also by the recurrence relations.

Theorem 2.3. *Let $n \geq 2$, $k \geq 2$ be integers. Then $g(k,n,0) = 1$, $g(k,n,1) = n-k+1$. Let $p \geq 2$. If $n < pk$, then $g(k,n,p) = 0$ and for $n \geq pk$ we have the recurrence relation $g(k,n,p) = g(k,n-1,p) + g(k,n-k,p-1)$.*

Proof. The initial conditions are obvious. Assume now that $n \geq pk$ and $p \geq 2$. Let $\mathcal{Y} \subset \mathcal{X}$. From the definition of the subfamily $\mathcal{Y}$ we have that it has exactly p subsets such that each subset Y contains k consecutive integers and each two subsets from $\mathcal{Y}$ are disjoint. Let $g_n(k,n,p)$ (respectively $g_{-n}(k,n,p)$) be the number of p-elements subfamilies $\mathcal{Y}$ such that there is $Y \in \mathcal{Y}$ and $n \in Y$ (respectively for each $Y \in \mathcal{Y}$, $n \notin Y$). Then it is clear that $g(k,n,p) = g_n(k,n,p) + g_{-n}(k,n,p)$.

We distinguish two cases:

1. There is $Y \in \mathcal{Y}$ such that $n \in Y$.

Then it is clear that $Y = \{n-k+1,...,n\}$ and for every $Y^* \in \mathcal{Y}$, $Y^* \neq Y$, $Y^* \cap Y = \emptyset$. Consequently $\mathcal{Y} = \mathcal{Y}' \cup Y$, where $\mathcal{Y}'$ is an arbitrary $(p-1)$-element subfamily satisfying the condition (ii) of the family $\mathcal{X}' = \{\{i,...,i+k-1\}; i = 1,...,n-2k+1\}$. Hence $g_n(k,n,p) = g(k,n-k,p-1)$.

2. For each $Y \in \mathcal{Y}$, $n \notin Y$.

In this case we deduce that we can find the subfamily $\mathcal{Y}$ in the family $\mathcal{X}'' = \{\{i,...,i+k-1\}; i = 1,...,n-k\}$. Hence $g_{-n}(k,n,p) = g(k,n-1,p)$.

In consequence from the above cases we obtain for the numbers $g(k,n,p)$ the k-th order linear recurrence on the form $g(k,n,p) = g(k,n-1,p) + g(k,n-k,p-1)$.

Thus the Theorem is proved. □

Theorem 2.4. *Let $n \geq 2$, $k \geq 2$ be integers. Then*

$G(k,n) = 1$, for $n \leq k-1$,

$G(k,n) = n-k+2$, for $k \leq n \leq 2k-1$ and

for $n \geq 2k$ we have $G(k,n) = G(k,n-1) + G(k,n-k)$

Proof. If $n = 2,...,k-1$, then $p = 0$. Hence $G(k,n) = g(k,n,0) = 1$. If $k \leq n \leq 2k-1$, then $p = 0$ or $p = 1$. So, $G(k,n) = \sum\limits_{p=0}^{1} g(k,n,p) = g(k,n,0) + g(k,n,1) = 1 + n - k + 1 = n - k + 2$. Assume now that $n \geq 2k$. Then by the Theorem 2.3 we have that

$$\begin{aligned} G(k,n) &= \sum_{p\geq 0} g(k,n,p) \\ &= g(k,n,0)+g(k,n,1)+\sum_{p\geq 2} g(k,n,p) \\ &= 1+n-k+1+\sum_{p\geq 2}(g(k,n-1,p)+g(k,n-k,p-1)) \\ &= n-k+2+\sum_{p\geq 2} g(k,n-1,p)+\sum_{p\geq 2} g(k,n-k,p-1) \\ &= 1+\sum_{p\geq 2} g(k,n-1,p)+1+n-k+\sum_{p\geq 1} g(k,n-k,p) \\ &= \sum_{p\geq 0} g(k,n-1,p)+\sum_{p\geq 0} g(k,n-k,p) \\ &= G(k,n-1)+G(k,n-k). \end{aligned}$$

Thus the Theorem is proved. □

From this Theorem and the relation (1) by simple observation we obtain

Corollary 2.5. *Let $n \geq k$, $k \geq 2$ be integers. Then $G(k,n) = F(k, n-(k-1))$.*

It is interesting that the numbers $G(k,n)$ (i.e. $F(k,n-(k-1))$) have also the graph interpretation with respect to the number of P_k-matchings of the graph P_n. The set X can be represented as the vertex set $V(P_n)$ of the graph P_n, where vertices from $V(P_n)$ are numbered by integers in the natural fashion. Consequently in the graph terminology for every i, $1 \leq i \leq n-k+1$ the subset $\{i, i+1, ..., i+k-1\}$ induces in P_n a component of a P_k-matching, for $k \geq 2$ and the number $F(k, n-(k-1))$, for $n \geq k-1$ is equal to the number of all P_k-matchings of the graph P_n. It is worth to mention that if $k=2$, then P_2-matching corresponds to matching and we obtain known result that the number of all matchings (known in the literature as the Hosoya index $Z(G)$ of a graph G) in P_n is equal to F_{n-1}, i.e. $G(2,n) = Z(P_n) = F_{n-1}$, see [3].

Let X be the set of n integers, $n \geq 3$, $k \geq 2$ and let $\mathcal{X}^* = \{\{i, i+1, ..., i+k-1\}; i = 1, ..., n-k+1\} \cup \{\{n-k+j+1, n-k+j+2, ..., j\}; j = 1, ..., k-1\}$. In the other words the family $\mathcal{X}^*$ contains subsets with k cyclically consecutive integers.

Let $\mathcal{Y}^* \subset \mathcal{X}^*$ be the subfamily of $\mathcal{X}^*$ such that

(iii) $|\mathcal{Y}^*| = p$ for fixed $p \geq 0$ and

(iv) for each $Y_t, Y_q \in \mathcal{Y}^*$, $t \neq q$ holds $Y_t \cap Y_q = \emptyset$.

Let $h(k,n,p)$ denote the number of all p-elements subfamilies $\mathcal{Y}$ and further let $H(k,n) = \sum_{p\geq 0} h(k,n,p)$.

Theorem 2.6. *Let $n \geq 3$, $k \geq 2$ be integers. Then $h(k,n,0) = 1$, $h(k,n,1) = n$. Let $p \geq 2$. If $n < pk$, then $h(k,n,p) = 0$ and for $n \geq pk$ we have the following relation $h(k,n,p) = g(k,n-1,p) + k \cdot g(k,n-k,p-1)$.*

Proof. The initial conditions are obvious. Let $n \geq pk$ and $p \geq 2$. Let $\mathcal{Y}^* \subset \mathcal{X}^*$. From the definition of the subfamily $\mathcal{Y}^*$ we have that the subfamily $\mathcal{Y}^*$ has exactly p subsets such that each subset of $\mathcal{Y}^*$ contains k cyclicaly consecutive integers and each two subsets from $\mathcal{Y}^*$ are disjoint.

Let $h_n(k,n,p)$ (respectively; $h_{-n}(k,n,p)$) be the number of p-elements subfamilies $\mathcal{Y}^*$ such that there is a subset $Y \in \mathcal{Y}^*$ such that $n \in Y$ (respectively; for each $Y \in \mathcal{Y}^*$ $n \notin Y$). Then $h(k,n,p) = h_n(k,n,p) + h_{-n}(k,n,p)$.

We consider two cases:

1. There is $Y \in \mathcal{Y}^*$ such that $n \in Y$.

Then it is clear that Y contains k cyclically consecutive integers and without loss on the generalization assume that $Y = \{n-k+1,...,n\}$. Then for every $Y' \in \mathcal{Y}^*$, $Y' \neq Y$ holds $Y \cap Y' = \emptyset$. Hence $\mathcal{Y}^* = \mathcal{Y}' \cup Y$ where $\mathcal{Y}'$ is an arbitrary $(p-1)$-elements subfamily of the family $\mathcal{X}' = \{\{i,...,i+k-1\}; i = 1,...,n-2k+1\}$. Using previous results for the number $g(k,n,p)$ we deduce that the number of subfamilies $\mathcal{Y}^*$ including the set Y is equal to $g(k,n-k,p-1)$. Because there are exactly k subsets $Y \in \mathcal{Y}^*$ including the integer n so $h_n(k,n,p) = k \cdot g(k,n-k,p-1)$.

2. For each $Y \in \mathcal{Y}^*$ $n \notin Y$.

In this case it is clear that to find the number of p-elements families $\mathcal{Y}^*$ it suffices to find this families in the family $\mathcal{X}' = \{\{i,...,i+k-1\}; i = 1,...,n-k\}$.

Using investigations from the Theorem 2.3 it immediately follows that $h_{-n}(k,n,p) = g(k,n-1,p)$.

Finally from the above cases we have that $h(k,n,p) = g(k,n-1,p) + k \cdot g(k,n-k,p-1)$, which completes the proof. □

Now we give a more comfortable identity for the numbers $h(k,n,p)$, for $n \geq pk$.

Theorem 2.7. *Let $k \geq 2$, $n \geq 3$, $p \geq 0$ be integers. Then for $n \geq pk$ we have the following relation $h(k,n,p) = h(k,n-1,p) + h(k,n-k,p-1)$.*

Proof. Let $k \geq 2$, $n \geq 3$, $p \geq 0$ be integers. Then using Theorem 2.6 we have that $h(k,n,p) = k \cdot g(k,n-k,p-1) + g(k,n-1,p) = k \cdot (g(k,n-2k,p-2) + g(k,n-k-1,p-1)) + g(k,n-k-1,p-1) + g(k,n-2,p) = k \cdot g(k,(n-k)-k,p-2) + g(k,(n-k)-1,p-1) + k \cdot g(k,(n-1)-k,p-1) + g(k,(n-1)-1,p) = h(k,n-k,p-1) + h(k,n-1,p)$

Hence $h(k,n,p) = h(k,n-1,p) + h(k,n-k,p-1)$, which completes the proof. □

Using result given in Theorem 2.1 we can give also a direct formula for the number $h(k,n,p)$.

Theorem 2.8. *Let $k \geq 2$, $n \geq 3$, $p \geq 0$ be integers. Then $h(k,n,p) = k\binom{n-1-(k-1)p}{p-1} + \binom{n-1-(k-1)p}{p}$.*

This equality follows immediately from Theorem 2.1 and Theorem 2.7, so we omit the proof.

Theorem 2.9. *Let $n \geq 3$, $k \geq 2$ be integers. Then*

$H(k,n) = 1$, *for* $n \leq k-1$,

$H(k,n) = n+1$, *for* $k \leq n \leq 2k-1$ *and*

for $n \geq 2k$ *we have* $H(k,n) = H(k,n-1) + H(k,n-k)$.

Proof. The initial conditions are obvious. Assume now that $n \geq 2k$. Then by Theorem 2.6 and Theorem 2.7 we obtain

$$\begin{aligned} H(k,n) &= \sum_{p\geq 0} h(k,n,p) \\ &= h(k,n,0) + h(k,n,1) + \sum_{p\geq 2} h(k,n,p) \\ &= 1 + n + \sum_{p\geq 2} (h(k,n-k,p-1) + h(k,n-1,p)) \\ &= 1 + n + \sum_{p\geq 1} h(k,n-k,p) + \sum_{p\geq 2} h(k,n-1,p) \\ &= n + \sum_{p\geq 0} h(k,n-k,p) + \sum_{p\geq 0} h(k,n-1,p) - 1 - (n-1) \\ &= \sum_{p\geq 0} h(k,n-k,p) + \sum_{p\geq 0} h(k,n-1,p) \\ &= H(k,n-k) + H(k,n-1), \end{aligned}$$

which ends the proof. □

Corollary 2.10. *Let $n \geq k$, $k \geq 2$ be integers. Then $H(k,n) = L(k,n)$.*

It is worth to mention that the numbers $H(k,n)$ (i.e. the numbers $L(k,n)$) have the graph interpretation with respect to the number of P_k-matchings of the graph C_n, $n \geq k$. It is easy to observe that the set X can be represented as the vertex set $V(C_n)$ of the graph C_n listing vertices from $V(C_n)$ in the natural way. Because every subset $\{i, i+1, ..., i+k-1\}$ induces in C_n a component of the P_k-matching, $k \geq 2$, so the number $L(k,n)$ is equal to the number of all P_k-matchings of the graph C_n. Clearly if $k = 2$, then we obtain known results that $H(2,n) = Z(C_n) = L_n$, see [3].

3. Main Results

Using previous results and their graph representations we can give the number of different subfamilies of the set of n integers. To do it firstly we consider the case which includes other possibilities.

Let $X = \{1, 2, ..., n\}$, $n \geq 2$ be the set of n integers. Let $k \geq 2$ and for a fixed t, $1 \leq t \leq n-k+1$, let $\mathcal{F}_t$ be the family of all 2-elements subsets of the subset $\{t, ..., t+k-1\}$. Let $\mathcal{F} = \{\mathcal{F}_t; t = 1, ..., n-k+1\}$. We define a subfamily $\mathcal{K}$ of the family $\mathcal{F}$ in the following way

(v) $|\mathcal{K}| = p$, for fixed $p \geq 0$

(vi) for each $\mathcal{F}_i, \mathcal{F}_j \in \mathcal{K}$ and $i \neq j$ holds $|i-j| \geq k$.

Let $\eta(k,n,p)$ be the number of all p-elements subfamilies $\mathcal{K}$. Then $\mathcal{N}(k,n) = \sum_{p\geq 0} \eta(k,n,p)$ is the number of all subfamilies $\mathcal{K}$.

To prove the result for the number $\mathcal{N}(k,n)$ we use graph tools and we need auxiliarily a special operation of graphs, namely a k-th power of a graph. Let $d_G(x,y)$ denote the distance between x and y in the graph G. For an integer $k \geq 2$, by the k-th power of a graph G we mean a graph G^k such that $V(G^k) = V(G)$ and $E(G^k) = \{xy; d_G(x,y) \leq k\}$.

Theorem 3.1. *Let $n \geq k$, $k \geq 2$ be integers. Then $\mathcal{N}(k,n) = F(k, n-(k-1))$.*

Proof. By previous considerations we know that for $n \geq k$ the number $F(k, n-(k-1))$ is the number of all P_k-matchings in a graph P_n. Using the operation of $(k-1)$-power of a graph P_n we obtain a graph P_n^{k-1} in which every subgraph K_k corresponds to subgraph P_k of P_n, where $k \leq n$. Moreover the definition of the family $\mathcal{K}$ gives that every element of $\mathcal{K}$ corresponds to complete subgraph K_k of P_k^{k-1}. In consequence the result follows by Corollary 2.5. □

Clearly in the graph interpretation the number $\mathcal{N}(k,n)$ is the number of K_k-matchings in a graph P_n^{k-1}. As the consequence of it and by previous results we have

Theorem 3.2. *Let $n \geq 2$, $k \geq 2$ be integers. Then $\eta(k,n,0) = 1$, $\eta(k,n,1) = n-k+1$. Let $p \geq 2$. If $n < pk$, then $\eta(k,n,p) = 0$ and for $n \geq pk$ we have the recurrence relation $\eta(k,n,p) = \eta(k,n-1,p) + \eta(k,n-k,p-1)$.*

Similarly we can consider the cyclic version of the family $\mathcal{F}$.

Let $X = \{1,2,...,n\}$, $n \geq 3$ be the set of n integers. Let $k \geq 2$ be an integer. For a fixed t, $1 \leq t \leq n-k+1$ we recall that the family $\mathcal{F}_t$ be the family of all 2-elements subsets of the subset $\{t,...,t+k-1\}$. For $n-k+2 \leq t \leq n$ the family $\mathcal{F}_t$ be the family of all 2-elements subsets of the subset $\{t,..,n,1,...,t-n+k-1\}$.

Let $\mathcal{F}^* = \{\mathcal{F}_t; t = 1,...,n-k+1, n-k+2,...,n\}$. We define a subfamily $\mathcal{K}^*$ of the family $\mathcal{F}^*$ in the following way:

(vi) $|\mathcal{K}^*| = p$, for fixed $p \geq 0$

(viii) for each $\mathcal{F}_i, \mathcal{F}_j \in \mathcal{K}^*$ and $i \neq j$ holds $k \leq |i-j| \leq n-k$.

Let $\mu(k,n,p)$ be the number of all p-elements subfamilies $\mathcal{K}^*$. Then $\mathcal{M}(k,n) = \sum_{p\geq 0} \mu(k,n,p)$ is the number of all subfamilies $\mathcal{K}^*$.

Proceeding analogously as in Theorem 3.1 we can prove

Theorem 3.3. *Let $n \geq k$, $k \geq 2$ be integers. Then $\mathcal{M}(k,n) = L(k,n)$.*

Analogously as for the number $\eta(k,n,t)$ we can immediately give

Theorem 3.4. *Let $n \geq 3$, $k \geq 2$ be integers. Then $\mu(k,n,0) = 1$, $\mu(k,n,1) = n$. Let $p \geq 2$. If $n < pk$, then $\mu(k,n,p) = 0$ and for $n \geq pk$ we have the following relation $\mu(k,n,p) = \mu(k,n-1,p) + \mu(k,n-k,p-1)$.*

For fixed t, $1 \leq t \leq n-k+1$ let $\overline{\mathcal{F}_t}$ be an arbitrary family of 2-elements subsets of the set $\{t,...,t+k-1\}$ such that for each distinct $u, v \in \{t,...,t+k-1\}$ there are in $\overline{\mathcal{F}_t}$ subsets $\{u,t_1\},\{t_1,t_2\},...,\{t_r,v\}$. Then we say that the subfamily $\overline{\mathcal{F}_t}$ has the property $\mathcal{C}$. Let $\overline{\mathcal{F}} = \{\overline{\mathcal{F}_t}; t = 1,...,n-k+1\}$. Let $\mathcal{P}$ be a subfamily of $\overline{\mathcal{F}}$ such that

(ix) $|\mathcal{P}| = p$, for fixed $p \geq 0$

(x) for each $\overline{\mathcal{F}_i}, \overline{\mathcal{F}_j} \in \overline{\mathcal{F}}$ such that $i \neq j$ holds $|i-j| \geq k$.

Let $r(k,n,p)$ be the number of all p-elements subfamilies P. Then $\mathcal{R}(k,n) = \sum\limits_{p \geq 0} r(k,n,p)$ is the number of all subfamilies $\mathcal{P}$.

Theorem 3.5. *Let $n \geq k$, $k \geq 2$ be integers. Then $\mathcal{R}(k,n) = F(k,n-(k-1))$.*

Proof. By Theorem 3.1 we obtain that $F(k,n-(k-1))$ is the number of all K_k-matchings of the graph P_n^{k-1}. The definition of the family $\mathcal{P}$ gives that it contains such subfamilies which in the graph interpretation have elements corresponding to a spanning subgraph of K_k in P_n^{k-1}. Consequently counting the complete subgraphs in P_n^{k-1} is equivalent to counting the arbitrary spanning subgraphs of K_k in P_n^{k-1}, replacing every subgraph K_k by one of it's spanning subgraphs, which ends the proof. □

As the consequence we have

Theorem 3.6. *Let $n \geq 2$, $k \geq 2$ be integers. Then $r(k,n,0) = 1$, $r(k,n,1) = n-k+1$. Let $p \geq 2$. If $n < pk$, then $r(k,n,p) = 0$ and for $n \geq pk$ we have the recurrence relation $r(k,n,p) = r(k,n-1,p) + r(k,n-k,p-1)$.*

The graph interpretation of the family $\mathcal{P}$ inspired us to consider the generalization of an H-matching of the graph G.

Let G be a graph. For a given collection $\mathcal{H} = H_1,...,H_m$ of graphs by an $\mathcal{H}$-matching $\mathfrak{M}$ of G we mean a family of subgraphs of G such that each connected component of $\mathfrak{M}$ is isomorphic to some H_i, $1 \leq i \leq m$. Moreover we put that the empty set is an $\mathcal{H}$-matching of G, too. It can see that if $H_i = H$, for all $i = 1,...,m$ then we obtain the definition of H-matching. Among $\mathcal{H}$-matchings it is interesting to study $\mathcal{H}$-matchings where H_i, $i = 1,...,m$ belong to the same class of graphs, for example if H_i, $i = 1,...,m$ are complete graphs $K_1,...,K_m$ or if H_i, $i = 1,...,m$ are of the same order.

Corollary 3.7. *Let $n \geq k$, $k \geq 2$ be integers. Let $\mathcal{H} = H_1,...,H_m$ be a collection of fixed spanning subgraphs of K_k. Then $\mathcal{R}(k,n)$ is the minimal number of all $\mathcal{H}$-matchings of P_n^{k-1}.*

Corollary 3.8. *Let $n \geq k$, $k \geq 2$ be integers. Let $\mathcal{H} = H_1, ..., H_s$ be a collection of all spanning subgraphs of K_k. Then $\mathcal{R}(k, n)$ is the minimal number of all $\mathcal{H}$-matchings of an arbitrary spanning subgraph P of P_n^{k-1}.*

Analogously we can give the main result for cyclic version of the family $\overline{\mathcal{F}}$.

We recall that for a fixed t, $1 \leq t \leq n-k+1$ let $\overline{\mathcal{F}_t}$ be the family of 2-elements subsets of the set $\{t, ..., t+k-1\}$ such that the subfamily $\overline{\mathcal{F}_t}$ has the property $\mathcal{C}$. For $n-k+2 \leq t \leq n$ let $\overline{\mathcal{F}_t}$ be the family of 2-elements subsets of the subset $\{t, ..., n, 1, ..., t-n+k-1\}$, with the property $\mathcal{C}$.

Let $\mathcal{F}^* = \{\overline{\mathcal{F}_t}; t = 1, ..., n\}$. Let $\mathcal{P}^*$ be a subfamily of $\overline{\mathcal{F}^*}$ such that

(xi) $|\mathcal{P}^*| = p$, for fixed $p \geq 0$

(xii) for each $\overline{\mathcal{F}_i}, \overline{\mathcal{F}_j} \in \mathcal{P}^*$ such that $i \neq j$ holds $k \leq |i-j| \leq n-k$.

Let $r^*(k, n, p)$ be the number of all p-elements subfamilies $\mathcal{P}^*$. Then $\mathcal{R}^*(k, n) = \sum_{p \geq 0} r^*(k, n, p)$ is the number of all subfamilies $\mathcal{P}^*$.

Proceeding analogously as in Theorem 3.5 we obtain

Theorem 3.9. *Let $n \geq k$, $k \geq 2$ be integers. Then $\mathcal{R}^*(k, n) = L(k, n)$.*

Analogously we can prove

Theorem 3.10. *Let $n \geq 3$, $k \geq 2$ be integers. Then $r^*(k, n, 0) = 1$, $r^*(k, n, 1) = n$. Let $p \geq 2$. If $n < pk$, then $r^*(k, n, p) = 0$ and for $n \geq pk$ we have the following relation $r^*(k, n, p) = r^*(k, n-1, p) + r^*(k, n-k, p-1)$.*

Corollary 3.11. *Let $n \geq k$, $k \geq 2$ be integers. Let $\mathcal{H} = H_1, ..., H_m$ be a collection of fixed spanning subgraphs of K_k. Then $R^*(k, n)$ is less or equal to the number of $\mathcal{H}$-matchings of a graph C_n^{k-1}.*

Corollary 3.12. *Let $n \geq k$, $k \geq 2$ be integers. Let $\mathcal{H} = H_1, ..., H_m$ be a collection of all spanning subgraphs of K_k. Then $R^*(k, n)$ is less or equal to the number of $\mathcal{H}$-matchings of an arbitrary spanning subgraph C of a graph C_n^{k-1}, where C includes C_n as a subgraph.*

References

[1] C. Berge, *Principles of combinatorics*, Academic Press New York and London, 1971.

[2] R. Diestel, *Graph theory*, Springer-Verlag, Heidelberg, New-York, Inc., 2005.

[3] I. Gutman and S. Wagner, Maxima and minima of the Hosoya index and the Merrifield-Simmons index. A survey of results and techniques, *Acta Appl. Math.*, **112/3** (2010), 323–346.

[4] M. Kwaśnik and I. Włoch, The total number of generalized stable sets and kernels of graphs, *Ars Combin.*, **55** (2000), 139–146.

[5] A. Włoch, I. Włoch, Generalized sequences and k-independent sets in graphs, *Discrete Appl. Math.*, **158** (2010), 1966–1970.

[6] A. Włoch, *Some identities for the generalized Fibonacci numbers and the generalized Lucas numbers* (Submitted).