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Subgraph Summability Number of Paths and Cycles*

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Abstract

A vertex labeling of a graph G is a mapping $\alpha : V(G) \rightarrow \mathbb{N}$, assigning a positive integer value to each vertex. With this we can consider labels of connected induced subgraphs $G[U]$ for $U \subseteq V(G)$, and define $\alpha(G[U]) = \sum_{u \in U} \alpha(u)$. The subgraph summability number of a connected graph G is the largest integer $\sigma(G)$ so that label sums of connected induced subgraphs cover the integers 1 through $\sigma(G)$ for some vertex labeling of G. We investigate subgraph summability labelings for paths and cycles and provide upper and lower bounds.

Keywords: graph labeling, subgraph summability number, paths, cycles, cyclic difference sets.

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1. Introduction

In this paper we only consider simple, connected graphs with no loops or multiple edges. Denote the vertex set of graph G by $V(G)$. Given a graph G, a subgraph H is said to be *induced* if for any pair of vertices x, y in $V(H)$, (x, y) is an edge of H if and only if (x, y) is an edge in G. We denote the number of connected induced subgraphs of a graph G as $c(G)$. Further details on graph theory can be found in [6].

A *vertex labeling* of a graph is a function $\alpha : V(G) \to \mathbb{N}$ which assigns a positive integer value to vertices of G. We call α an N-labeling if for every integer $1 \leq n \leq N$, there exists a connected induced subgraph H of G such that $\alpha(H) = \sum_{x \in V(H)} \alpha(x) = n$. It is useful to notice that, by definition, any N-labeling is also an $(N-k)$-labeling for $0 \leq k < N$. The *subgraph summability number* of a graph G is the largest integer N for which there is an N-labeling of G and is denoted $\sigma(G)$.

We describe the relationship between the subgraph summability number and the number of connected induced subgraphs using redundancy and excess. Suppose α is an N-labeling and let t_i denote the number of connected induced subgraphs whose label sum is i for each $i \leq N$. The *redundancy* of a labeling α is $r(\alpha) = \sum_{i=1}^{N} (t_i - 1)$, the number of connected induced subgraphs with repeated label sums less than or equal to N. The *excess* of a labeling α is $e(\alpha) = \sum_{m>N} t_m$, the number of connected induced subgraphs whose label sum is greater than N. Notice that for any N-labeling α, $c(G) = N + r(\alpha) + e(\alpha)$. By definition $\sigma(G) \leq c(G)$. Moreover, if α has no excess and no redundancy, then $\sigma(G) = c(G)$, and α is said to be a *sharp labeling* of G.

There is previous work on subgraph summability number from a variety of sources. Subgraph summability number was mentioned in Doug West's open problems column under the name of subgraph sums [7]. The column mentions some known results at that time. Stephen Penrice first used the term "subgraph summability number" in [3]. In [3], Penrice gave upper and lower bounds for the subgraph summability number of paths and cycles. He also gave explicit formulas for $\sigma(G)$ when G is a complete graph, a complete graph minus one edge, or a star graph. The subgraph summability number for bicliques was given explicitly in [2]. The question of subgraph summability of bicliques was also studied under the terminology of IC-coloring [4], [5].

2. Paths

A path on n vertices, denoted P_n, is a connected graph where all but two vertices have degree two. We label the vertices of P_n as v_i for $1 \leq i \leq n$. Furthermore, let (v_i, v_{i+1}) be an edge of P_n for $i < n$.

Penrice [3] researched the subgraph summability number of paths. He found upper and lower bounds for the subgraph summability of paths, which are summarized below. We include proofs of Penrice's results for the sake of completeness.

Proposition 2.1. [3] *Let P_n be the path on n vertices. Then*

$$\sigma(P_n) \geq \begin{cases} \frac{n^2+6n-4}{4}, & n \text{ even} \\ \frac{n^2+6n-3}{4}, & n \text{ odd} \end{cases}$$

Proof. Penrice's labeling varies with parity of n, so each case is considered separately. First, suppose n is even. Label the vertices of P_n as follows:

- $\alpha(v_i) = 1$ for $1 \leq i \leq \frac{n}{2}$,
- $\alpha(v_i) = \frac{n}{2} + 2$ for $\frac{n}{2} < i < n$,
- $\alpha(v_n) = \frac{n}{2} + 1$.

We wish to obtain a connected induced graph with label sum of m for each $1 \leq m \leq \frac{n^2+6n-4}{4}$. Suppose $m = j(\frac{n}{2} + 2) + r$ where $r \leq \frac{n}{2} + 1$ and $j \leq \frac{n}{2} - 1$.

If $r < \frac{n}{2} + 1$ then use the connected induced subgraph containing the vertices, $v_{\frac{n}{2}+1}$ through $v_{\frac{n}{2}+j}$ and vertices $v_{\frac{n}{2}-r+1}$ through $v_{\frac{n}{2}}$. This is a connected induced subgraph with label sum m.

If $r = \frac{n}{2} + 1$ then use the connected induced subgraph containing vertices, v_{n-j} through v_{n-1} and vertex v_n.

In order to get label sums of $m = (\frac{n}{2} - 1)(\frac{n}{2} + 2) + (\frac{n}{2} + 1) + r$ where $r \leq \frac{n}{2}$, use the connected induced subgraph with vertices $v_{\frac{n}{2}-r+1}$ through v_n. Therefore, the graph has connected induced subgraphs with label sums up to $(\frac{n}{2} - 1)(\frac{n}{2} + 2) + (\frac{n}{2} + 1) + \frac{n}{2} = \frac{n^2+6n-4}{4}$ as desired.

Now consider n odd. Label the vertices of P_n as follows:

- $\alpha(v_i) = 1$ for $1 \leq i \leq \left(\frac{n-1}{2}\right)$
- $\alpha(v_i) = \left(\frac{n-1}{2}\right) + 2$ for $\left(\frac{n-1}{2}\right) < i < n$
- $\alpha(v_n) = \left(\frac{n-1}{2}\right) + 1$

The proof of this case is similar to when n is even and we have $\sigma(P_n) \geq \frac{n^2+6n-3}{4}$ as desired. □

Penrice also showed that there are no sharp labelings for paths on more than three vertices. We repeat his proof and use it to establish an upper bound for $\sigma(P_n)$, since by virtue of there being no sharp labeling, the subgraph summability number must be less than the number of connected induced subgraphs of P_n.

Proposition 2.2. [3] *There is no sharp labeling for P_n if $n > 3$.*

Proof. Suppose P_n is a path on $n > 3$ vertices. The number of connected induced subgraphs of P_n is $c(P_n) = \binom{n+1}{2} = \sum_{i=1}^{n} i$.

Suppose there is an $\binom{n+1}{2}$-labeling α for P_n. Then α is a sharp labeling and has no redundancy and no excess. Hence the n vertices must be labeled from the set $\{1, 2, \ldots, n\}$. Also, there must be connected induced subgraphs with label sums $\binom{n+1}{2} - 1$ and $\binom{n+1}{2} - 2$. Therefore the end vertices of the path must be labeled 1 and 2. Assume without loss of generality that $\alpha(v_1) = 1$ and $\alpha(v_n) = 2$.

If $\alpha(v_2) < n$, then $\alpha(v_1) + \alpha(v_2) = \alpha(v_k)$ for some k thus α has a redundancy.

If $\alpha(v_2) = n$ then $\alpha(v_{n-1}) < n$ when $n \geq 4$. Therefore, $\alpha(v_{n-1}) + \alpha(v_n) \leq n + 1$, which implies α has a redundancy. □

We now create a new labeling which provides an improvement on Penrice's lower bound for $\sigma(P_n)$ found in Proposition 2.1.

Proposition 2.3. *If P_n is a path on n vertices, then $\sigma(P_n) \geq \frac{n^2+7n-10}{4}$.*

Proof. Consider P_n and choose k as follows:

- For $n \equiv 0 \pmod 4$ let $k = \left(\frac{n-2}{2}\right)$
- For $n \equiv 1 \pmod 4$ let $k = \left(\frac{n-1}{2}\right)$
- For $n \equiv 2 \pmod 4$ let $k = \left(\frac{n}{2}\right)$
- For $n \equiv 3 \pmod 4$ let $k = \left(\frac{n-3}{2}\right)$

Then label the vertices of P_n as follows:

- $\alpha(v_2) = \alpha(v_3) = \ldots = \alpha(v_{k+1}) = 2$
- $\alpha(v_{k+2}) = \alpha(v_{k+4}) = \ldots = \alpha(v_{n-1}) = 1$
- $\alpha(v_{k+3}) = \alpha(v_{k+5}) = \ldots = \alpha(v_{n-2}) = 2k + 4$
- $\alpha(v_n) = 2k + 2$
- $\alpha(v_1) = 1 + \sum_{i=k+2}^{n} \alpha(v_i)$

Assume we are in the case $n \equiv 0 \pmod 4$, $n \geq 8$. We wish to obtain connected induced subgraphs with label sums of m for each $1 \leq m \leq \frac{n^2+7n-8}{4}$. Begin by writing $m = j(2k + 5) + r$ where $r \leq 2k + 4$ and $j \leq \frac{n}{4} - 1$. We consider the following cases.

(i) If $r = 2s$ where $s \leq k$, then use the connected induced subgraph containing vertices v_{k+2} through v_{k+2j+1} and vertices v_{k-s+2} through v_{k+1}.

(ii) If $r = 2s + 1$ where $s \leq k$, then use the connected induced subgraph containing vertices v_{k+2} through v_{k+2j+1}, vertices v_{k-s+2} through v_{k+1}, and vertex v_{k+2j+2}.

(iii) If $r = 2k + 2$, then use the connected induced subgraph containing vertices v_{n-2j} through v_{n-1} and vertex v_n.

(iv) If $r = 2k + 3$, then use the connected induced subgraph containing vertices v_{n-2j} through v_{n-1}, vertex v_n, and also vertex v_{n-2j-1}.

(v) If $r = 2k + 4$, then use the connected induced subgraph containing vertices v_{n-2} through v_{n-2j-2}.

This system works for all values of m up to $\sum_{i=k+2}^{n} \alpha(v_i)$. If m is larger than this sum, write $m = \sum_{i=k+2}^{n} \alpha(v_i) + 1 + 2s$ or $m = \sum_{i=k+2}^{n} \alpha(v_i) + 2s$ for $s \leq k$.

If $m = \sum_{i=k+2}^{n} \alpha(v_i) + 1 + 2s$ then the connected induced subgraph containing vertices v_1 through v_{s+1} has the desired label sum.

If $m = \sum_{i=k+2}^{n} \alpha(v_i) + 2s$ then the connected induced subgraph containing vertices v_{k+2-s} through v_n has the desired label sum.

Therefore we may obtain any m from 1 to $\sum_{i=k+2}^{n} \alpha(v_i) + 1 + 2k = \binom{\frac{n^2+7n-8}{4}}{}$.

The proofs for $n \equiv 1, 2, 3 \pmod 4$ are similar. Below are the results of the proofs for all four cases:

- $n \equiv 0 \pmod 4$ then $\sigma(P_n) \geq \left(\frac{n^2+7n-8}{4}\right)$
- $n \equiv 1 \pmod 4$ then $\sigma(P_n) \geq \left(\frac{n^2+7n-8}{4}\right)$
- $n \equiv 2 \pmod 4$ then $\sigma(P_n) \geq \left(\frac{n^2+7n-10}{4}\right)$
- $n \equiv 3 \pmod 4$ then $\sigma(P_n) \geq \left(\frac{n^2+7n-10}{4}\right)$

□

Remark 2.4. *The lower bound in Proposition 2.3 is better than the lower bound in Proposition 2.1 for $n \geq 8$ and becomes increasingly better for larger n. The difference between the two lower bounds grows approximately as $\frac{n}{4}$. However, the ratio of each of the lower bounds to $c(P_n)$ has a limit of $\frac{1}{2}$.*

3. Cycles

A cycle on n vertices, denoted C_n, is a connected graph where each vertex has degree two. We label the vertices of C_n as v_i for $1 \leq i \leq n$ and let (v_i, v_{i+1}) be an edge of C_n for $1 \leq i \leq n-1$. We let (v_n, v_1) be the remaining edge of C_n.

One can easily check that $c(C_n) = n(n-1)+1$. A lower bound for the subgraph summability number of cycles was established in [3]. We provide a proof for the sake of completeness.

Proposition 3.1. [3] *Let C_n be the cycle on n vertices. Then* $\left(\frac{n^2+4n-7}{2}\right) \leq \sigma(C_n) \leq n(n-1)+1$.

Proof. It is easy to verify the claim for $n = 3$, so we will assume $n \geq 3$ and consider C_{n+1}. We label the first n vertices as in Proposition 2.1 and label the last vertex with one more than the sum of the labels on vertices of P_n.

- $\alpha(v_i) = 1$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $\alpha(v_i) = \left\lfloor \frac{n}{2} \right\rfloor + 2$ for $\left\lfloor \frac{n}{2} \right\rfloor < i < n$
- $\alpha(v_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$
- $\alpha(v_{n+1}) = \begin{cases} \frac{n^2+6n}{4}, & n \text{ even} \\ \frac{n^2+6n+1}{4}, & n \text{ odd} \end{cases}$

Suppose that n is even. We show that α is a $\frac{n^2+6n-2}{2}$-labeling for C_{n+1}. From Proposition 2.1 it is clear we have label sums from 1 through $\frac{n^2+6n-4}{4}$. It suffices to show that for any $\frac{n^2+6n}{4} \leq m \leq \frac{n^2+6n-2}{2}$ there is a connected induced subgraph with label sum m.

Since n is even $\alpha(v_{n+1}) = \frac{n^2+6n}{4}$. For any m between $\frac{n^2+6n}{4}$ and $\frac{n^2+6n-2}{2}$ write $m = \frac{n^2+6n}{4} + j\left(\frac{n}{2}+2\right) + r$ where $0 \leq r \leq \frac{n-2}{2}$ and $j \leq \frac{n}{2}$.

If $r = j = 0$, then use the connected induced subgraph containing vertex v_{n+1}.

If $j = 0$ and $1 \leq r < \frac{n}{2}$, then use the connected induced subgraph with vertex v_{n+1} and vertices v_1 through v_r.

If $j > 0$ and $r = 0$, then use the connected induced subgraph with vertex v_{n+1}, vertex v_1, and vertices v_{n-j+1} through v_n.

If $j > 0$ and $r > 0$, then use the connected induced subgraph with vertex v_{n+1}, vertices v_1 through v_{r+1}, and vertices v_{n-j+1} through v_n.

With this system, we have found connected induced subgraphs with label sums for any $m \leq \sum_{i=1}^{n+1} \alpha(v_i) = \frac{n^2+6n-2}{2}$. This means that $\sigma(C_{n+1}) \geq \frac{n^2+6n-2}{2}$ and $\sigma(C_n) \geq \frac{n^2+4n-7}{2}$. The proof is similar when n is odd. □

We now apply the path labeling shown in Proposition 2.3 to cycles to establish an improved lower bound for $\sigma(C_n)$.

Proposition 3.2. *Let C_n denote the cycle on n vertices. Then* $\left(\frac{n^2+5n-16}{2}\right) \leq \sigma(C_n) \leq n(n-1)+1$.

Proof. Label C_n with the same labeling as P_n in Proposition 2.3. We give a proof for the case $n \equiv 0 \pmod 4$ where k is $\left(\frac{n-2}{2}\right)$. According to Proposition 2.3, we have label sums up to $\left(\frac{n^2+7n-8}{4}\right)$.

To obtain label sums of m for $\frac{n^2+7n-8}{4} \leq m \leq \frac{n^2+5n-10}{2}$, write $m = (\frac{n}{4}-1)(2k+5) + (2k+3) + j(2k+5) + r$ for $r \leq 2k+4$ and $j \leq \frac{n}{4} - 1$. We take the following cases.

(i) If $r = 0$, then use the connected induced subgraph containing vertices v_1 through v_{k+2} and vertices v_{k+3} through v_{k+2j+2}.

(ii) If $r = 1$, then use the connected induced subgraph containing vertex v_n, vertex v_1, and vertices v_{n-1} through v_{n-2j}.

(iii) If $r = 2$, then use the connected induced subgraph containing vertex v_n, vertex v_1, vertices v_{n-1} through v_{n-2j}, and vertex v_{n-2j-1}.

(iv) If $r = 2 + 2s$ for $1 \leq s \leq k$, then use the connected induced subgraph containing vertex v_n, vertex v_1, vertices v_{n-1} through v_{n-2j}, and vertices v_2 through v_{s+1}.

(v) If $r = 2 + 2s + 1$ for $1 \leq s \leq k$, then use the connected induced subgraph containing vertex v_n, vertex v_1, vertices v_{n-1} through v_{n-2j}, vertices v_2 through v_{s+1}, and vertex v_{n-2j-1}.

(vi) If $r = 2k+4$, then use the connected induced subgraph containing vertices v_1 through v_{k+3} and vertices v_{k+4} through v_{k+2j+3}.

Doing this gives label sums of graphs from 1 to $\sum_{i=1}^{n} \alpha(v_i)$. Thus we get the subgraph label sums up to the following:

$$\left(\frac{n^2+11n+4}{4}\right) + \left(\frac{n}{4} - 2\right) + \left(\frac{n}{4} - 1\right)(2k+4) = \left(\frac{n^2+5n-10}{2}\right)$$

Below are the results of the proofs for all four cases:

- $n \equiv 0 \pmod 4$ then $\sigma(C_n) \geq \left(\frac{n^2+5n-10}{2}\right)$
- $n \equiv 1 \pmod 4$ then $\sigma(C_n) \geq \left(\frac{n^2+5n-12}{2}\right)$

- $n \equiv 2 \pmod 4$ then $\sigma(C_n) \geq \binom{\frac{n^2+5n-16}{2}}{}$
- $n \equiv 3 \pmod 4$ then $\sigma(C_n) \geq \binom{\frac{n^2+5n-12}{2}}{}$

□

Remark 3.3. *The lower bound in Proposition 3.2 is better than the lower bound in Proposition 3.1 for cycles on four or more vertices. The bound becomes increasingly better for cycles on more vertices. The difference between the two lower bounds grows approximately as $\frac{n}{2}$. This difference is better than the difference for paths shown in Remark 2.4. The ratio of each of the lower bounds for the cycles to $c(C_n)$ has a limit of $\frac{1}{2}$. This is the same ratio as in the case for paths.*

Unlike paths which have no sharp labeling if $n \geq 4$, there are cycles which have $\sigma(C_n) = c(C_n)$ for large n. Sharp labelings for cycles on 3, 4, 5, 6, 8, 9, and 10 vertices were given in [3]. In [3] it is claimed that $\sigma(C_7) = 39 < c(C_7)$. We have confirmed this claim with an exhaustive computer search which tested all possible labelings of C_7 with vertex labels up to 45.

Penrice's sharp labelings of cycles are examples of a more widely known result. It was mentioned in West's column that there is a sharp labeling of C_n when $n-1$ is a prime power. We give a proof which arises from projective planes and cyclic difference sets.

Definition 3.4. *A (v, k, λ) difference set is a set $D = \{d_i : 1 \leq i \leq k\}$ in a group of order v (written additively), so that each nonidentity element of the group is represented λ times in the set $\{d_i - d_j : i \neq j\}$.*

It is well known (for example see [1]) that when $n-1$ is a prime power, there is a $(n^2-n+1, n, 1)$ difference set in the cyclic group of order n^2-n+1 (addition modulo n^2-n+1). The difference set can be thought of as one line of a projective plane of order $n-1$. We utilize this difference set in labeling the vertices of C_n.

Theorem 3.5. *If $n-1$ is a prime power, then $\sigma(C_n) = c(C_n)$.*

Proof. Denote the $(n^2-n+1, n, 1)$ difference set D by $\{d_i : 1 \leq i \leq n\}$ and assume $d_i < d_{i+1}$. Define the labeling α by $\alpha(v_i) = (d_{i+1} - d_i) \bmod (n^2-n+1)$ for $i < n$ and $\alpha(v_n) = (n^2-n+1) + d_1 - d_n$.

To show that α is a $c(C_n)$ labeling, first notice that the sum of all labels is (n^2-n+1). To get the other integers, we utilize the difference set. For each k from 1 to n^2-n, there exists a pair (j, l) such that $d_j - d_l \equiv k \bmod (n^2-n+1)$.

If $j > l$, then $\sum_{i=l}^{j-1} \alpha(v_i) = \sum_{i=j}^{l} (d_{i+1} - d_i) = d_j - d_l$. This is k since $d_j > d_l$.

If $l > j$, then $\sum_{i=l}^{n} v_i + \sum_{i=1}^{j-1} v_i = (n^2-n+1) + d_j - d_l$. This is k since $d_j - d_l$ is k mod (n^2-n+1) and $d_l > d_j$.

Therefore each k, from 1 to n^2-n+1 is represented by a subgraph of C_n. □

We leave as open questions if there are sharp labelings on cycles when $n-1$ is not a prime power and if the labelings we have given above can be improved.

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