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On Independent and (d, n)-domination numbers of hypercubes

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Abstract

In this paper we consider the (d, n)-domination number, $\gamma_{d,n}(Q_n)$, the distance-d domination number $\gamma_d(Q_n)$ and the connected distance-d domination number $\gamma_{c,d}(Q_n)$ of n-dimensional hypercube graphs Q_n. We show that for $2 \le d \le \lfloor n/2 \rfloor$, and $n \ge 4$, $\gamma_{d,n}(Q_n) \le 2^{n-2d+2}$, improving the bound of Xie and Xu [19]. We also show that $\gamma_d(Q_n) \le 2^{n-2d+2-r}$, for $2^r - 1 \le n - 2d + 1 < 2^{r+1} - 1$, and $\gamma_{c,d}(Q_n) \le 2^{n-d}$, for $1 \le n - d + 1 \le 3$, and $\gamma_{c,d}(Q_n) \le 2^{n-d-1} + 4$, for $n - d + 1 \ge 4$.

Moreover, we give an upper bound of the independent domination number, $\gamma_i(Q_n)$ and the total domination number, $\gamma_t(Q_n)$ of Q_n. We show that $\gamma_i(Q_n) \le 2^{n-k}$, $\gamma_t(Q_n) \le 2^{n-k}$ for $2^k - 1 < n < 2^{k+1} - 1$ and $k \ge 1$ also we show that $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$ when $n = 2^k$ and $k \ge 3$.

Keywords: independent domination number, (d, k)-domination number, hypercubes.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *open neighborhood* $N(v)$ of a vertex $v \in V$ is the set $N(v) = \{u \in V | uv \in E\}$. Every vertex $u \in N(v)$ is called a *neighbor* of v. A subset $D \subseteq V$ is called a *dominating set* if every vertex $v \in V - D$ has at least one neighbor $u \in D$. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G.

A dominating set D is called an independent dominating set if no two vertices of D are adjacent. The *independent domination number* $\gamma_i(G)$ of a graph G equals the minimum cardinality of a independent dominating set in G. A dominating set D is called a perfect dominating set if every vertex $v \in V(G) - D$ has exactly one neighbor in D. The *perfect domination number* $\gamma_p(G)$ of a graph G equals the minimum cardinality of a perfect dominating set in G. A dominating set D is called a total dominating set if for every $v \in V(G)$,

$d(v, D - \{v\}) = 1$. The *total domination number* $\gamma_t(G)$ of a graph G equals the minimum cardinality of a total dominating set in G. A *connected dominating set* is a dominating set D whose induced subgraph $G[D] = (D, E \cap (D \times D))$ is a connected graph. The *connected domination number* $\gamma_c(G)$ equals the minimum cardinality of a connected dominating set in G.

A set $D \subseteq V$ is a *distance-d dominating set* if every vertex $v \in V - D$ lies on a path of length at most d to some vertex $u \in D$. The *distance-d domination number* $\gamma_d(G)$ equals the minimum cardinality of a distance-d dominating set in G, and the *connected distance-d domination number* $\gamma_{c,d}(G)$ equals the minimum cardinality of a connected distance-d dominating set in G.

The *Cartesian product* $G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \Box G_2 = (V_1 \times V_2, E_1 \Box E_2)$, whose vertex set is the Cartesian product $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \Box G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$. The *hypercube* Q_n, for $n \geq 1$, is defined recursively as follows: $Q_1 = P_2$, the path consisting of two adjacent vertices; and $Q_n = Q_{n-1} \Box P_2$. Equivalently, the n-cube Q_n is defined as the graph whose vertex set consists of all binary vectors $u = u_1 u_2 \ldots u_n$ of length n, and two vertices are adjacent if and only if the corresponding vectors differ in exactly one coordinate. For any two vertices $u, v \in V(Q_n)$, the distance between them is defined as $d(u, v) = \Sigma_{i=1}^{n} |u_i - v_i|$. Hypercubes Q_n have many attractive and useful properties, for example, they have 2^n vertices, they are bipartite (all cycles have even length), they are n-regular (every vertex has n neighbors), they are n-connected (between any two vertices there are n (internally) vertex disjoint paths, and they have diameter n (the maximum distance between any two vertices is n) etc (cf. Laborde and Hebbare [8], Li et al [12], Mane and Waphare [13]). Hypercubes are also one of the most popular interconnection networks for parallel computation, cf. Leighton [10].

Relatively little is known about the domination numbers $\gamma(Q_n)$ of hypercubes. Exact values have only been determined for $n \leq 8$, cf. Arumugam and Kala [1], Mollard [14], and Stanton and Kalbfleisch [15], and for $n = 2^k - 1$ and $n = 2^k$, which follows from coding theory, cf. Berlekamp [2], Cohen, Honkala, Litsyn and Lobstein [4], Hamming [5], Jha [7], Thompson [16] and Van Wee [17]. It is obvious that $\gamma(Q_n) \leq 2^{n-k}$, for $2^k - 1 \leq n < 2^{k+1} - 1$ and $k \geq 1$.

In 1997 Li and Xu [11] defined the (d,n)-*dominating number* in n-connected graphs, in order to more accurately characterize the reliability of networks. Let $G = (V, E)$ be an n-connected graph, let S be a non-empty and proper subset of V, and let y be a vertex in $V - S$. For a given positive integer d, y is (d,n)-*dominated by* S if there are at least n internally vertex disjoint $y - S$ paths in G, each of which is of length at most d. A set S is a (d,n)-*dominating set* if every vertex $v \in V - S$ is (d,n)-dominated by S. The (d,n)-*dominating number*, denoted $\gamma_{d,n}(G)$ equals the minimum cardinality of a (d,n)-dominating set in G.

Li and Xu [11] discovered several general properties of (d,n)-dominating sets and determined the value of $\gamma_{d,n}(G)$ in various families of n-connected graphs. In particular they showed that $\gamma_{d,n}(Q_n) = 2$ for $n \geq 2$ and $n-1 \leq d \leq n$. In 2002, Lu and Zhang [3] showed that $\gamma_{d,n}(Q_n) = 2$ for $n \geq 4$ and $\lfloor n/2 \rfloor + 2 \leq d \leq n$. In 2007, Xie and Xu [19] showed that $\gamma_{1,n}(Q_n) = 2^{n-1}$, for $n > 1$, $\gamma_{d,n}(Q_n) = 2$, for $d = \lfloor n/2 \rfloor + 2 \leq d \leq n$ and $n > 2$, and $\gamma_{d,n}(Q_n) \leq 2^{n-d+1}$, for $d \leq \lfloor n/2 \rfloor$ and $n > 3$.

In this paper we sharpen the upper bound in the last case above by showing that $\gamma_{d,n}(Q_n) \leq 2^{n-2d+2}$, for $2 \leq d \leq \lfloor n/2 \rfloor$ and $n \geq 4$. We also show that (i) $\gamma_d(Q_n) \leq 2^{n-2d+2-r}$, for $2^r - 1 \leq n - 2d + 1 < 2^{r+1} - 1$, (ii) $\gamma_{c,d}(Q_n) \leq 2^{n-d}$, for $1 \leq n - d + 1 \leq 3$, and (iii) $\gamma_{c,d}(Q_n) \leq 2^{n-d-1} + 4$, for $n - d + 1 \geq 4$.

Harary and Livingston observed in [6] that $\gamma(Q_n) = \gamma_i(Q_n)$ holds when $n \in \{1, 2, 3, 4, 6\}$, but for $n = 5$ we have $\gamma(Q_5) = 7$ and $\gamma_i(Q_5) = 8$. They asked whether $\gamma(Q_n) = \gamma_i(Q_n)$ for any $n > 7$. Van Wee [17] proved that $\gamma(Q_n) = \gamma_t(Q_n)$ for $n = 2^k$. In this paper, we show that $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$ when $n = 2^k$ and $k \geq 3$. In fact, we give an upper bound of the independent domination number, $\gamma_i(Q_n)$ and the total domination number, $\gamma_t(Q_n)$ of Q_n. We show that $\gamma_i(Q_n) \leq 2^{n-k}$, $\gamma_t(Q_n) \leq 2^{n-k}$ for $2^k - 1 < n < 2^{k+1} - 1$ and $k \geq 1$. For any terms not defined here, the reader is referred to West [18].

2. On the (d, n)-dominating numbers of hypercubes Q_n

For any d, $1 \leq d \leq n-1$, we can write $Q_n = Q_d \Box Q_{n-d}$. Let y be the $(0,1)$-vector of length $n-d$ corresponding to a vertex in $V(Q_{n-d})$, and let $Q_{d.y}$ denote the subgraph of Q_n induced by the vertices the last $n-d$ coordinates of which equal the vector y. Similarly, let x be the $(0,1)$-vector of length d corresponding to a vertex in $V(Q_d)$, and let $Q_{x.n-d}$ denote the subgraph of Q_n induced by the vertices the first d coordinates of which equal the vector x.

Note that $Q_{d.y}$ is isomorphic to Q_d, and $Q_{x.n-d}$ is isomorphic to Q_{n-d}. Furthermore,

(i) $\bigcup_{y \in \{0,1\}^{n-d}} V(Q_{d.y}) = V(Q_n)$,

(ii) for any $y_1, y_2 \in \{0,1\}^{n-d}$, $V(Q_{d.y_1}) \cap V(Q_{d.y_2}) = \emptyset$,

(iii) $\bigcup_{x \in \{0,1\}^d} V(Q_{x.n-d}) = V(Q_n)$,

(iv) for any $x_1, x_2 \in \{0,1\}^d$, $x_1 \neq x_2$, $V(Q_{x_1.n-d}) \cap V(Q_{x_2.n-d}) = \emptyset$,

(v) for any $x \in \{0,1\}^d$ and for any $y \in \{0,1\}^{n-d}$, $E(Q_{x.n-d}) \cap E(Q_{d.y}) = \emptyset$,

(vi) $\cup_{y \in \{0,1\}^{n-d}} E(Q_{d.y}) \bigcup \cup_{x \in \{0,1\}^d} E(Q_{x.n-d}) = E(Q_n)$.

Let y_1 and y_2 be any two adjacent vertices in Q_{n-d}. It follows that for any vertex $w_1 \in V(Q_{d.y_1})$ there exists a unique vertex $w_2 \in V(Q_{d.y_2})$ such that w_1 is adjacent to w_2 in Q_n, in which case we say that vertices w_1 and w_2 are a *corresponding pair* of vertices in Q_n. Since vertex y_1 has degree $n-d$ in Q_{n-d}, it follows that every vertex $w_1 \in V(Q_{d.y_1})$ has $n-d$ corresponding vertices in Q_n.

Let the 2^n vertices of Q_n be labelled $A_0, A_1, \ldots, A_{2^n-1}$. Let Let y be the $(0,1)$-vector of length $n-d$ corresponding to an arbitrary vertex in $V(Q_{n-d})$ and let $\{A_{m.y} | 0 \leq m \leq 2^d - 1\}$ denote the 2^d vertices of $Q_{d.y}$, each of which has the same $n-d$ last coordinates. Let $A_{0,y}$ and $A_{1,y}$ denote the two vertices of Q_n whose first d components are all 0's and all 1's, respectively, that is $A_{0,y} = (0,0,\dots,0,y)$ and $A_{1,y} = (1,1,\dots,1,y)$, respectively. Now, for any vertex $A_j \in Q_n$, there exists a unique y (the last $n-d$ coordinates of A_j) such that $A_j = A_{m,y}$, for some $m \in \{0,1\}^d$.

Theorem 1. *For any positive integer n,*

(I) $\gamma_{d,n}(Q_n) \le 2^{n-2d+2}$, *for $n \ge 4$ and $2 \le d \le \lfloor n/2 \rfloor$.*

(II) $\gamma_d(Q_n) \le 2^{n-2d+2-r}$, *for $n \ge 6$ and $2 \le d < \lfloor n/2 \rfloor$, where $2^r - 1 \le n - 2d + 1 \le 2^{r+1} - 1$.*

(III) $\gamma_{c,d}(Q_n) \le 2^{n-d}$, *for $1 \le n-d+1 \le 3$ and $\gamma_{c,d}(Q_n) \le 2^{n-d-1}+4$, for $n-d+1 \ge 4$.*

Proof. In [19], Xie and Xu showed that

$$\gamma_{d,n}(Q_n) = 2, \text{ for } d = \lfloor n/2 \rfloor + 1 \text{ and } n > 2, \tag{1}$$

where the set $S = \{(0,0,\dots,0),(1,1,\dots,1)\}$ is a minimum cardinality (d,n)-dominating set.

For $d \ge 2$, we assume that $n = 2d-1$ for n odd, and $n = 2d-2$ for n even. In both cases $\lfloor n/2 \rfloor = d-1$.

Let $A_i \in V(Q_n)$, where $0 \le i \le 2^n - 1$, and let $A_0 = (0,0,\dots,0)$ and $A_1 = (1,1,\dots,1)$. For any $j \notin \{0,1\}$, we can see that either $d(A_j, A_0) \le d = \lfloor n/2 \rfloor + 1$ or $d(A_j, A_1) \le d$ in Q_n. Without loss of generality, let $d(A_j, A_0) \le d$. Also, $d(A_0, A_1) = n$ in Q_n. Therefore, for any $A_j \in V(Q_n)$, $j \notin \{0,1\}$, if $d(A_j, A_0) = f$, for some $1 \le f \le d$, then $d(A_j, A_1) = n - f$. Thus, either $d(A_j, A_0) \le d-1$, or $d(A_j, A_1) \le d-1$ in Q_n, where

$$n = 2d-1 \text{ for } n \text{ odd, or } n = 2d-2 \text{ for } n \text{ even.} \tag{2}$$

(I): For $n \ge 4$ and $2 \le d \le \lfloor n/2 \rfloor$, split Q_n into 2^{n-2d+1} vertex disjoint copies of Q_{2d-1}. For any vertex $y \in V(Q_{n-2d+1})$ label the vertices in the y copy of Q_{2d-1} as $Q_{2d-1.y}$. Let $A_{0.y}$ and $A_{1.y}$ be the vertices of $Q_{2d-1.y}$ whose last $n-2d+1$ coordinates form the vector y and whose first $2d-1$ coordinates are all 0, or all 1, respectively. Let $S_y = \{A_{0.y}, A_{1.y}\}$.

Claim. $S = \cup_{y \in V(Q_{n-2d+1})} S_y$ is a (d,n)-dominating set of Q_n.

Choose any vertex A_j of $Q_n - S$. There exists a unique y such that $A_j = A_{m.y} \in V(Q_{2d-1.y})$, where $j \notin \{0,1\}$. Using (1) above, we can observe that $\gamma_{d,2d-1}(Q_{2d-1}) = 2$, for any $d \ge 2$, which means that there exist $2d-1$ vertex disjoint paths from $A_{m.y}$ to S_y, each of which has length at most d in $Q_{2d-1.y}$.

As $A_{m.y}$ corresponds to exactly $n-(2d-1)$ vertices which are not in $V(Q_{2d-1.y})$, without loss of generality let $A_{m.y_i}$ denote the corresponding vertices of $A_{m.y}$, where $A_{m.y} = \{A_{m.y_i} \in V(Q_{2d-1.y_i}) | y_i \ne y, i \in \{\sigma(1), \sigma(2), \dots, \sigma(n-(2d-l))\}\}$, where σ is a permutation in S_{n-2d-1}.

We know that for $d \ge 2$ and $j \notin \{0,1\}$, either $d(A_{m.y}, A_{0.y}) \le d-1$, or $d(A_{m.y}, A_{1.y}) \le d-1$ in $Q_{2d-1.y}$ (see (2)).

Without loss of generality suppose $d(A_{m.y}, A_{0.y}) \leq d-1$ in $Q_{2d-1.y}$, and let P_{y_i} be a shortest path from $A_{m.y_i}$ to $A_{0.y_i}$.

We can then construct a path P_i of length d from $A_{m.y}$ to S_{y_i} by starting at vertex $A_{m.y}$ going along the edge to vertex $A_{m.y_i}$ and then using the path P_{y_i}.

Note that these paths P_i are pairwise vertex disjoint.

Thus, the $2d-1$ vertex disjoint paths from $A_{m.y}$ to S_y in $Q_{2d-1.y}$, together with the $n-(2d-1)$ paths in the set $\{P_i | i \in \{\sigma(1), \sigma(2), \ldots, \sigma(n-(2d-l))\}\}$ gives n pairwise vertex disjoint paths from A_j to S, each of which has length at most d.

But $|S| = 2^{n-2d+1} \times 2 = 2^{n-2d+2}$, which shows that $\gamma_{d,n}(Q_n) \leq 2^{n-2d+2}$, for $2 \leq d \leq \lfloor n/2 \rfloor$ and $n \geq 4$.

(II): For $n \geq 6$ and $2 \leq d < \lfloor n/2 \rfloor$, let $Q_n = Q_{2d-1} \Box Q_{n-2d+1}$, and for any $y \in V(Q_{n-2d+1})$, let $Q_{2d-1.y}$ denote the subgraph of Q_n induced by the vertices whose last $n-2d+1$ coordinates form the vector y.

Further, for any $x \in V(Q_{2d-1})$, let $Q_{x.n-2d+1}$ denote the subgraph of Q_n induced by the vertices, the first $2d-1$ coordinates of which form the vector x.

Let $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$, where $x_0, x_1 \in V(Q_{2d-1})$.

It follows immediately that $A_{0.y} \in (Q_{2d-1.y} \cap Q_{x_0.n-2d+1})$ and $A_{1.y} \in (Q_{2d-1.y} \cap Q_{x_1.n-2d+1})$, for $y \in V(Q_{n-2d+1})$, where $A_{0.y}$ and $A_{1,y}$ are vertices in $Q_{2d-1.y}$ whose last $n-2d+1$ coordinates form the vector y and the remaining first $2d-1$ coordinates are all 0, or all 1, respectively.

We know that for any positive integer r such that $2^r - 1 \leq n-2d+1 < 2^{r+1}-1$, $\gamma(Q_{n-2d+1}) \leq 2^{n-2d+1-r}$.

Let D_0 and D_1 denote minimum dominating sets of $Q_{x_0.n-2d+1}$ and $Q_{x_1.n-2d+1}$, respectively.

Claim. $\gamma_d(Q_n) \leq |D_0| + |D_1|$.

Let A_j be any vertex in $V(Q_n) - (D_0 \cup D_1)$.

If $A_j = A_{0.y}$ or $A_j = A_{1,y}$, for any $y \in V(Q_{n-2d+1})$, then $d(A_{0,y}, D_0) \leq 1$ and $d(A_{1,y}, D_0) \leq 1$.

For $j \notin \{0,1\}$, there exists a unique y such that $A_j = A_{m.y} \in V(Q_{2d-1.y})$, and either $d(A_{m.y}, A_{0.y}) \leq d-1$ or $d(A_{m.y}, A_{1.y}) \leq d-1$ in $Q_{2d-1.y}$ (see (2) above).

Therefore, $d(A_{m.y}, D_0) \leq d$ or $d(A_{m.y}, D_1) \leq d$. Thus, $\gamma_d(Q_n) \leq |D_0| + |D_1| \leq 2^{n-2d+2-r}$.

(III): For $n \geq 2$ and $2 \leq d \leq n$, let $Q_n = Q_{d-1} \Box Q_{n-d+1}$, and for any $y \in V(Q_{n-d+1})$, let $Q_{d-1.y}$ denote the subgraph of Q_n induced by the vertices whose last $n-d+1$ coordinates form the vector y.

Further, for any vertex $x \in V(Q_{d-1})$, let $Q_{x.n-d+1}$ denote the subgraph of Q_n induced by the vertices whose first $d-1$ coordinates form the vector x.

Let $x_0 = (0, 0, \ldots, 0) \in V(Q_{2d-1})$.

It follows immediately that $A_{0,y} \in Q_{2d-1.y} \cap Q_{x_o.n-2d+1}$, for $y \in V(Q_{n-2d+1})$, where $A_{0,y}$ is the vertex in $Q_{d-1.y}$ whose last $n-d+1$ coordinates form the vector y and the remaining first $d-1$ coordinates are all 0.

Let D_c denote a minimum connected dominating set of $Q_{x_0.n-d+1}$.

In [1] Arumugam et al proved that the connected domination number of Q_n, $\gamma_c(Q_n) \leq 2^{n-1}$, for $1 \leq n \leq 3$ and $\gamma_c(Q_n) \leq 2^{n-2}$ for $n \geq 4$.

Claim. $\gamma c, d(Q_n) \leq |D_c|$.

Let A_j be any vertex in $Q_n - D_c$. If $A_j = A_{0.y}$ or $A_j = A_{1.y}$ for any $y \in V(Q_{n-d+1})$, then $d(A_{0.y}) \leq 1$ and $d(A_{1.y}, A_{0.y}) = d - 1$.

For $j \notin \{0.1\}$, there exists a unique y such that $A_j = A_{m.y} \in V(Q_{d-1.y})$ and $d(A_{m.y}, A_{0.y}) \leq d-1$ in $Q_{d-1.y}$, and therefore $d(A_{m.y}, D_c) \leq d$.

Thus, $\gamma_{c,d}(Q_n) \leq |D_c| \leq 2^{n-d}$, for $1 \leq n-d+1 \leq 3$, and $\gamma_{c,d}(Q_n) \leq |D_c| \leq 2^{n-d+1} + 4$, for $n - d + 1 \geq 4$. □

3. On the independent dominating numbers of hypercubes Q_n

Lee [9] proved the following lemma.

Lemma 1. [9] *Let $X = \{x_1, x_2, ..., x_n\}$ be a symmetric generating set for a group A, and let S be an independent perfect dominating set of the Cayley graph $C(A, X)$. Then :*

(a) for each $i = 1, 2, ..., n$, Sx_i is an independent perfect dominating set of $C(A, X)$ and

(b) $\{S, Sx_1, ..., Sx_n\}$ forms a vertex partition of $C(A, X)$.

Note that, for $A = Z_2 \times Z_2 \times ...... \times Z_2(n \text{ times})$, $X = 10...0, 010...0, , 00...01$ is a symmetric generating set of A and hence the Cayley graph $C(A, X)$ is isomorphic to the hypercube Q_n (see [9]). Moreover, we can easily observe that for independent perfect dominating set D, any two vertices $u, v \in D$ are such that $d(u, v) \geq 3$.

We know that $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$ for $n = 2^k - 1$, $k \geq 1$. Now we prove our main Theorem of this section.

Theorem 2. *For any positive integer n, let k be an integer such that $2^k - 1 < n < 2^{k+1} - 1$. Then 2^{n-k} is an upper bound for both the independent domination number and the total domination number of Q_n. Moreover $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$ for $n = 2^k$ and $k \geq 1$.*

Proof. For $n \geq 1$, there always exists $k \geq 1$ such that $2^k - 1 \leq n < 2^{k+1} - 1$. For $n = 2^k - 1$ $(k \geq 1)$ we know $\gamma(Q_n) = \gamma_i(Q_n)$.

So assume $2^k - 1 < n < 2^{k+1} - 1$ $(k \geq 1)$ and decompose Q_n as $Q_n = Q_d \times Q_{n-d}$ where $d = 2^k - 1$. As a hypercube is bipartite, we partition $V(Q_{n-d})$ into two sets say

X and Y such that $X \cup Y = V(Q_{n-d})$ and $X \cap Y = \phi$. Now we fix perfect independent dominating set of Q_d say D_1 and construct another perfect independent dominating set say D_2 where $D_2 = \{u + e_1 : u \in D_1, e_1 = (1, 0, 0, ..., 0)\}$ therefore $D_1 \bigcap D_2 = \phi$ (see Lemma 1). It is known that $|D_1| = |D_2| = 2^{d-k}$. For $1 \leq j \leq 2$, let $(D_j, r) = \{(u, r) : u \in D_j, r \in V(Q_{n-d})\}$. For $r \in X$ we choose the dominating set (D_1, r) for $Q_{d.r}$ and for $r \in Y$ we choose the dominating set (D_2, r) for $Q_{d.r}$. As $D_1 \bigcap D_2 = \phi$, we observe that $D = (\cup_{r\in X}(D_1, r)) \bigcup (\cup_{r\in Y}(D_2, r))$ is an independent dominating set of Q_n. Also $|D| = (2^{n-d}) \times (2^{d-k})$, therefore $\gamma_i(Q_n) \leq 2^{n-k}$.
Now for $r \in V(Q_{n-d})$, if we consider (D_1, r) as a dominating set for $Q_{d.r}$ then it is easy to observe that $D = \cup_{r\in V(Q_{n-d})}(D_1, r)$ is a total dominating set of Q_n. Thus $\gamma_t(Q_n) \leq 2^{n-k}$ for $2^k - 1 < n < 2^{k+1} - 1$ and $k \geq 1$.

For $n = 2^k$ $(k \geq 1)$, Van Wee [17] proved that $\gamma(Q_n) = \gamma_t(Q_n) = 2^{n-k}$ and we know that $\gamma(Q_n) \leq \gamma_i(Q_n)$, hence $\gamma(Q_n) = \gamma_i(Q_n) = 2^{n-k}$. □

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