AKCE Int. J. Graphs Comb., 9, No. 2 (2012), pp. 187-193

Automorphism group of the Comellas-Fiol digraphs

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Communicated by: S. Arumugam
Received 28 December 2009; accepted 04 April 2012

Abstract

In 1995, Comellas and Fiol proposed a construction of record large vertex-transitive digraphs of given degree and diameter out of relatively small vertex-transitive input digraphs. We determine the automorphism group of the output digraph for a special but rather wide class of input digraphs.

Keywords: digraph, degree-diameter problem, automorphism group.

2010 Mathematics Subject Classification: 05C20, 05C12.

1. Introduction

The degree-diameter problem for digraphs is determination of the largest order $n(\Delta; D)$ of a digraph of maximum out-degree Δ and diameter at most D. For brevity, we refer to the history and recent development in the degree-diameter problem to the survey paper [6] and state here just the very basics. By the directed Moore bound we have

$$n(\Delta, D) \leq 1 + \Delta + \Delta^2 + \ldots + \Delta^D \tag{1}$$

which, however, is known to be sharp only if $\Delta = 1$ or $D = 1$. For $D = 2$ we have the upper bound

$$n(\Delta, 2) \leq \Delta + \Delta^2 \ , \tag{2}$$

known to be sharp and attained, for $\Delta \geq 3$, by exactly the line digraphs of complete digraphs [3]. It follows that all the extremal graphs for out-degree $\Delta \geq 3$ and diameter $D = 2$ are vertex-transitive, motivating thus a study of large vertex-transitive digraphs in the degree-diameter problem. Another reason for considering vertex-transitive examples in this context is computer search, where focusing on vertex-transitive and Cayley digraphs has the obvious reason of simplification of diameter testing.

The best currently known constructions of large vertex-transitive digraphs of given out-degree Δ and diameter D for rather general Δ and D are due to Comellas and Fiol [1] and Gómez [4].

The Comellas-Fiol construction, given in three variations in [1], produces 'large' vertex-transitive digraphs from certain 'small' input digraphs. Since the order of the output digraph depends on properties of the input digraph and on further two numerical parameters, the construction does not yield an explicit formula for the order of output digraphs depending solely on Δ and D.

The construction of Gómez [4] produces vertex-transitive digraphs of degree Δ, diameter D, and order $N = (\Delta + \lfloor D - 1/2 \rfloor)!/(\Delta - \lceil D + 1/2 \rceil)!$ for each $D \geq 3$ and $\Delta \geq \lceil \frac{D+1}{2} \rceil$. This construction has an ancestor in the work [2] which was further extended in [5] and finally superseded by [4].

In the case of vertex-transitive digraphs it is often useful to determine the (full) automorphism group of such a digraph. This was done in [7] in the case of the Faber-Moore-Chen digraphs [2] and it enabled to classify those Faber-Moore-Chen digraphs that are Cayley digraphs. In this contribution we will study only the first of the three constructions of Comellas and Fiol [1] since it serves as a basis for the other two construction (which can be analyzed analogously). Our goal is to determine the automorphism group of the Comellas-Fiol output digraphs for a special but rather wide class of input digraphs. We begin with introducing details of the first construction of [1].

Let Γ be a digraph of out-degree Δ, with vertex set V and dart set E, which will serve as input for the construction. Let t, k be positive integers; we do not let t appear in the forthcoming notation. The output digraph $CF(\Gamma, k)$ of (the first) construction of Comellas and Fiol is defined as follows. The vertex set of $CF(\Gamma, k)$ consists of elements of the form $(j|p_0p_1 \ldots p_{k-1})$ with $j \in \mathbb{Z}_{kt}$ and $p_i \in V$. All darts in $CF(\Gamma, k)$ have the form

$$(j|p_0p_1 \ldots p_{j-1}\ u\ p_{j+1} \ldots p_{k-1}) \to (j+1|p_0p_1 \ldots p_{j-1}\ v\ p_{j+1} \ldots p_{k-1}),$$

where $(u, v) \in E$. In order to obtain the desired output we need Γ to be vertex-transitive and r-reachable, which means that for every pair of vertices $u, v \in V$ there exists a (directed) walk of exactly r darts from u to v in Γ.

The main result of [1] states that if Γ is vertex-transitive, Δ-regular, r-reachable and has order N, then the resulting digraph $CF(\Gamma, k)$ is vertex-transitive, Δ-regular, of order ktN^k and diameter at most $k(r+t)-1$.

In fact, there is an obvious group acting vertex-transitively on $CF(\Gamma, k)$. Let $G = Aut(\Gamma)$ be the automorphism group of the input digraph Γ and let the cyclic group $\mathbb{Z}_{kt} = \{0, 1, \ldots, kt-1\}$ act on the direct product , $G^k = G \times \ldots \times G$ (k-times) by

$$(g_0, g_1, \ldots, g_{k-1}) \mapsto (g_s, g_{s+1}, \ldots, g_{s+k-1})$$

for any $s \in Z_{kt}$ where each subscript, after adding s, is reduced mod k in $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. Then the corresponding semidirect product $H = G^k \rtimes \mathbb{Z}_{kt}$ acts on vertices of the output digraph $CF(\Gamma, k)$ by the following rule: If $\alpha = (g_0, g_1, \ldots, g_{k-1}; s) \in H$, then

$$\alpha(j|p_0, p_1, \ldots, p_{k-1}) = (j - s|g_0(p_s)g_1(p_{s+1})g_2(p_{s+2}) \ldots g_{k-1}(p_{s+k-1})) .$$

It is easy to verify that this action of H makes the digraph $CF(\Gamma, k)$ vertex-transitive; note that $|H| = kt|G|^k$.

In what follows we prove that, in the case when $t = 1$, the group H is actually isomorphic to the (full) automorphism group of $CF(\Gamma, k)$ provided that the input digraph Γ satisfies some extra conditions.

2. The Result

First, let us introduce notions required for our proof. Let us fix a digraph Γ. An *alternating walk* from u to v in Γ is a walk (in which darts are allowed to be traversed in both directions) that contains no directed sub-walk of length two. Without explicitly specifying the digraph in the notation, let $R^+_\gamma(v)$ be the set of vertices of Γ to which there is an alternating walk of length γ from v that begins with a dart from v. Let $R^-_\gamma(v)$ be the set of vertices of Γ to which there is an alternating walk of length γ from v that begins with a dart terminating at v. We further say that a vertex u is γ-alternately reachable from v if $u \in R^+_\gamma(v) \cap R^-_\gamma(v)$. Denote the set of vertices that are γ-alternately reachable from v by $R_\gamma(v)$. Obviously, $R_\gamma(v) = R^+_\gamma(v) \cap R^-_\gamma(v)$. A digraph Γ is γ-alternately reachable if there is a number γ such that for every vertex v the set $R_\gamma(v) = V$. Note that we do not need γ to be minimal.

Example 1. *One of the families of γ-alternately reachable digraphs are digraphs constructed from γ-reachable graphs (introduced in the previous section) such that every edge (u, v) is replaced by two darts (u, v) and (v, u).*

Example 2. *Let $V = \mathbb{Z}_n$. If n is odd a digraph Γ that contains all darts of type $(i, i+1)$ and $(i, i-1)$ is γ-alternately reachable for all $\gamma \geq \frac{n+1}{2}$. If n is even a digraph Γ that contains all darts of type $(i, i+1)$ and $(i, i-1)$ and $(i, i+2m)$ for some m such that $1 \leq m < \frac{n}{2}$ is γ-alternately reachable for every $\gamma \geq 2m + 1 + \frac{n}{2}$.*

We use the following notation. The set $[(0|z_0 \ldots z_{k-1})]$ where every $z_l \in \mathbb{Z}_n$ contains all vertices of type $(0|z_0 \ldots z_{k-1})$. For example, the set of all vertices that have zero on the i-th position is denoted by $[(j|z_0 \ldots z_{i-2}0_{i-1}z_i \ldots z_{k-1})]$ where every $z_l \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_k$.

Lemma 1. *Let the input digraph Γ be $\{\gamma, \gamma+1, \gamma+2\}$-alternately reachable and vertex-transitive. Let σ be an automorphism of $CF(\Gamma, k)$ such that $\sigma(0|0 \ldots 0) = (0|0 \ldots 0)$. Then σ preserves the set $[(j|z_0 \ldots z_{i-1}0_i z_{i+1} \ldots z_{k-1})]$, where $i, j \in \mathbb{Z}_k$ and $\forall l,\ z_l \in \mathbb{Z}_n$.*

Proof. In $CF(\Gamma, k)$ there are only darts such that $(i|z_0 \ldots z_{k-1}) \to (i+1|b_0 \ldots b_{k-1})$, and hence a distance from $(0|0 \ldots 0)$ to $(i|z_0 \ldots z_{k-1})$ is $i + jk$ for some $j \in \mathbb{N}_0$. Consequently, the automorphism σ maps vertices as follows $\sigma(i|z_0 \ldots z_{k-1}) = (i|y_0 \ldots y_{k-1})$ where z_l is an arbitrary element from $\mathbb{Z}_n$ and $y_0, \dots, y_{k-1}$ are such that $(i|y_0 \dots y_{k-1})$ is an image of $(i|z_0 \dots z_{k-1})$. Roughly speaking σ preserves "i".

We will use some special walks from a fixed vertex v or from v and v' if $\sigma(v) = v'$. Denote by $W_{(a,b,c)_\gamma}(v)$ the set of all vertices such that there is a special walk from v: we start with the directed walk from v of length a (allowing $a = 0$) then we change the direction and go γ-times b steps (we change the direction after every b steps), finaly we go c steps. Note that if $W_{(1,1,0)_{\gamma-1}}(v) = W_{(0,1,0)_\gamma}(v) = V$ for every vertex $v \in V$, then Γ is γ-alternately reachable.

We will use the following two facts. If a vertex v is fixed by an automorphism σ, then σ preserves the set $W_{(a,b,c)_\gamma}(v)$. If $\sigma(v) = v'$, then σ maps the set $W_{(a,b,c)_\gamma}(v)$ to $W_{(a,b,c)_\gamma}(v')$.

First we look at the vertices of type $(i|z_0 \dots z_{i-1}0_i z_{i+1} \dots z_{k-1})$, i.e. the set of all vertices that have zero on the $(i+1)$-th position.

From definition of $CF(\Gamma, k)$ we have $W_{(1,0,0)_0}(0|0\dots0) = [(1|z_0 0 \dots 0)]$ where $z_0 \in R_1^+(0)$ in Γ. The set $[(1|z_0 0 \dots 0)]$ contains all out-neighbors of the vertex $(0|0\dots0)$ in $CF(\Gamma, k)$. Using the same reasoning we have

$$
\begin{aligned}
W_{(2,0,0)_0}(0|0\dots0) &= [(2|z_0 z_1 0 \dots 0)] \text{ where } z_0, z_1 \in R_1^+(0) \\
W_{(i,0,0)_0}(0|0\dots0) &= [(i|z_0 \dots z_{i-1} 0 \dots 0)] \text{ where } z_0, \dots, z_{i-1} \in R_1^+(0) \\
W_{(i,k-2,0)_1}(0|0\dots0) &= [(i+2|z_0 \dots z_{i-1} 0_i 0_{i+1} z_{i+2} \dots z_{k-1})] \\
&\qquad \text{where } z_0, \dots, z_{i-1} \in R_2^+(0) \text{ and } z_{i+2}, \dots, z_{k-1} \in R_1^-(0) \\
W_{(i,k-1,0)_1}(0|0\dots0) &= [(i+1|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})] \\
&\qquad \text{where } z_0, \dots, z_{i-1} \in R_2^+(0) \text{ and } z_{i+1}, \dots, z_{k-1} \in R_1^-(0) \\
W_{(i,k-1,0)_2}(0|0\dots0) &= [(i|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})] \\
&\qquad \text{where } z_0, \dots, z_{i-1} \in R_3^+(0) \text{ and } z_{i+1}, \dots, z_{k-1} \in R_2^-(0)
\end{aligned}
$$

For γ even, we obtain $W_{(i,k-1,0)_\gamma}(0|0\dots0) = [(i|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$, where $z_0, \dots, z_{i-1} \in R_{\gamma+1}^+(0)$ and $z_{i+1}, \dots, z_{k-1} \in R_\gamma^-(0)$. Hence σ preserves the set $[(i|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$, where $z_0, \dots, z_{i-1} \in R_{\gamma+1}^+(0)$ and $z_{i+1}, \dots, z_{k-1} \in R_\gamma^-(0)$. Since Γ is γ and $\gamma+1$ alternately reachable we have $R_{\gamma+1}^+(0) = R_\gamma^-(0) = V$ and hence σ preserves $[(i|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$ for all $z_l \in \mathbb{Z}_n$.

Moreover, for every $i, j \in \mathbb{Z}_k$ such that $i \geq j$ we have $W_{(i,k-1,j)_\gamma}(0|0\dots0) = [(i - j|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$, where $z_0, \dots, z_{i-j-1} \in R_{\gamma+1}^+(0)$, $z_{i-j}, \dots, z_{i-1} \in R_{\gamma+2}^+(0)$ and $z_{i+1}, \dots, z_{k-1} \in R_\gamma^-(0)$. For $\forall i, j \in \mathbb{Z}_k$ such that $i < j$ we have $W_{(i,k-1,j)_\gamma}(0|0\dots0) = [(i - j|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$, where $z_0, \dots, z_{i-1} \in R_{\gamma+2}^+(0)$ and $z_{i+1}, \dots, z_{i-j-1} \in R_\gamma^-(0)$ and $z_{i-j}, \dots, z_{k-1} \in R_{\gamma+1}^-(0)$. Since Γ is $\{\gamma, \gamma+1, \gamma+2\}$-alternately reachable we have for all j and i that σ preserves $[(j|z_0 \dots z_{i-1} 0_i z_{i+1} \dots z_{k-1})]$.

For γ odd we need walks $W_{(i,k-1,j)_\gamma}(0|0\dots0)$ where $i, j \in \mathbb{Z}_k$. □

Lemma 2. *Let the input digraph Γ be $\{\gamma,\gamma+1,\gamma+2\}$-alternately reachable and vertex-transitive with the vertex set $V=\mathbb{Z}_n$. Let σ be an automorphism of $CF(\Gamma,k)$ such that $\sigma(0|0\dots0)=(0|0\dots0)$. Then the automorphism σ must be of type $\sigma(i|z_0\dots z_{k-1}) = (i|\varphi_0(z_0)\dots\varphi_{k-1}(z_{k-1}))$ where $\varphi_j:\mathbb{Z}_n\to\mathbb{Z}_n$ is a bijection such that $\varphi_j(0)=0$ for $\forall i,j\in\mathbb{Z}_k$.*

Proof. From Lemma 1 we know that σ preserves $[(i|0\dots0x_{i-1}0\dots0)]$ where $x_{i-1}\in\mathbb{Z}_n$ and we have the following assumption for every i:

$$\sigma(i|0_0\dots0_{i-2}x_{i-1}0_i\dots0_{k-1}) = (i|0_0\dots0_{i-2}\varphi_{i-1}(x_{i-1})0_i\dots0_{k-1}),$$

where φ_{i-1} is a bijection from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ such that $\varphi_{i-1}(0)=0$.

Now we proceed similarly as in Lemma 1. Instead of the fixed vertex $(0|0\dots0)$ we have vertices v_l and v_l' such that $\sigma(v_l)=v_l'$.

For every γ, $W_{(k-1,k-1,0)_{\gamma-1}}(i|0\dots0x_{i-1}0\dots0) = [(i|z_0\dots z_{i-2}x_{i-1}z_i\dots z_{k-1})]$ where each $z_l\in R^+_\gamma(0)$, hence σ preserves $[(i|z_0\dots z_{i-2}x_{i-1}z_i\dots z_{k-1})]$ where each $z_l\in R^+_\gamma(0)=V$. Moreover, for γ even and every i and j we have

$$W_{(k-1,k-1,j)_{\gamma-1}}(i|0\dots0x_{i-1}0\dots0) = [(i+j|z_0\dots z_{i-2}x_{i-1}z_i\dots z_{i+j}\dots z_{k-1})]$$

where $z_i,\dots,z_{i+j-1}\in R^+_{\gamma+1}(0)=V$ and $z_0\dots z_{i-2},z_{i+j},\dots,z_{k-1}\in R^+_\gamma(0)=V$.

The same for $W_{(k-1,k-1,j)_{\gamma-1}}(i|0\dots0\varphi_{i-1}(x_{i-1})0\dots0)$. Hence we have σ maps the set

$$[(j|z_0\dots z_{i-2}x_{i-1}z_i\dots z_{k-1})] \text{ to } [(j|y_0\dots y_i\varphi_{i-1}(x_{i-1})y_i\dots y_{k-1})].$$

Hence the automorphism σ such that $\sigma(0|0\dots0) = (0|0\dots0)$ must be of type $\sigma(i|z_0\dots z_{k-1}) = (i|\varphi_0(z_0)\dots\varphi_{k-1}(z_{k-1}))$ where $\varphi_j:\mathbb{Z}_n\to\mathbb{Z}_n$ is a bijection such that $\varphi_j(0)=0$ for all $i,j\in\mathbb{Z}_k$. □

Lemma 3. *Let the input digraph Γ be $\{\gamma,\gamma+1,\gamma+2\}$-alternately reachable and vertex-transitive with the vertex set $V=\mathbb{Z}_n$. Let σ be an automorphism of $CF(\Gamma,k)$ such that $\sigma(0|0\dots0)=(0|0\dots0)$. Then the automorphism σ must be of type $\sigma(i|z_0\dots z_{k-1}) = (i|\varphi_0(z_0)\dots\varphi_{k-1}(z_{k-1}))$ where $\varphi_j:\mathbb{Z}_n\to\mathbb{Z}_n$ is an automorphism of Γ such that $\varphi_j(0)=0$.*

Proof. If φ_l is a bijection but not an automorphism of Γ then there is a dart $(u,v)\in E$ such that $(\varphi_l(u),\varphi_l(v))\notin E$. Hence σ such that $\sigma(i|z_0\dots z_{k-1}) = (i|\varphi_0(z_0)\dots\varphi_{k-1}(z_{k-1}))$, where φ_j is not an automorphism of Γ, maps a dart $((j|z_0\dots u_j\dots z_{k-1}),(j+1|z_0\dots v_j\dots z_{k-1}))$ to $((j|\varphi_0(z_0)\dots\varphi_j(u_j)\dots\varphi_{k-1}(z_{k-1})),(j+1|\varphi_0(z_0)\dots\varphi_j(v_j)\dots\varphi_{k-1}(z_{k-1})))$ but this is not a dart of $CF(\Gamma,k)$. □

Having made all the preparatory steps we are now ready to prove our main result.

Theorem 1. *If the input digraph Γ is $\{\gamma,\gamma+1,\gamma+2\}$-alternately reachable and vertex-transitive, then the order of the automorphism group of the Comellas-Fiol digraph $CF(\Gamma,k)$ is $|Aut(CF(\Gamma,k))| = k|G|^k$. Consequently, $Aut(CF(\Gamma,k))\cong[G]^k\rtimes\mathbb{Z}_k$.*

Proof. To determine the order of the automorphism group of vertex-transitive digraph it suffices to find the size of the stabilizer of a vertex v, i.e. the number of automorphisms that map the vertex v to itself. Particularly, we use $|\mathrm{Aut}(CF(\Gamma,k))| = |V(CF(\Gamma,k))| \cdot |\mathrm{Stab}(v)|$ where v is arbitrary vertex of $CF(\Gamma,k)$ and $V(CF(\Gamma,k))$ is the vertex set of $CF(\Gamma,k)$. We set $v = (0|0\ldots 0)$.

From Lemma 3 we have that there are at most $[\mathrm{Stab}_0(\Gamma)]^k$ of $CF(\Gamma,k)$ such that $\sigma(0|0\ldots 0) = (0|0\ldots 0)$, and then $|\mathrm{Stab}(0|0\ldots 0)| \leq [\mathrm{Stab}_0(\Gamma)]^k$. Since $\mathrm{Aut}(CF(\Gamma,k))$ is transitive, we have $|\mathrm{Aut}(CF(\Gamma,k))| = |V(CF(\Gamma,k))| \cdot |\mathrm{Stab}(0|0\ldots 0)| \leq k|V|^k[\mathrm{Stab}_0(\Gamma)]^k = k|G|^k$. □

We have the following immediate consequence of the main result.

Corollary 1. *The automorphism group of the Comellas-Fiol output digraphs $CF(\Gamma,k)$, where Γ is as in Example 1 or Example 2 and $t=1$, is isomorphic to $H = G^k \rtimes \mathbb{Z}_k$.* □

3. Conclusion

We have determined the full automorphism group of Comellas-Fiol digraphs in the case when $t=1$ and the input digraph is $\{\gamma,\gamma+1,\gamma+2\}$-alternately reachable. It would be interesting to find the automorphism group of the output digraph under weaker restrictions on the input digraphs, or for $t \geq 2$. Computer experiments, however, show that in such cases the automorphism group H of the output digraph may be bigger than the group from our main result; in particular, we may have $|H| > kt|G|^k$.

Example 3. *Consider the digraph Γ_1 with vertex set $V(\Gamma_1) = \mathbb{Z}_6$ and dart set $\{(i,i+1),(i,i+2);\ i \in Z_6\}$. Let the automorphism group of the digraph Γ_1 and of the corresponding output digraph $CF(\Gamma_1,k)$ for the parameters $k=t=2$ be G_1 and H_1, respectively. With the help of a computer we have $|G_1| = 6$ and $|H_1| = 576$, thus our hypothesis fails as $|H_1| = 576 > 2\cdot 2\cdot |G_1|^2 = 144$.*

It would be also interesting to determine the full automorphism group of the Gómez digraphs [4, 5]. Such type of results, including our Theorem 1, could be useful in classification of which of the large vertex-transitive digraphs for the degree-diameter problem are Cayley graphs, as was done in [7] for the Faber-Moore-Chen digraphs.

Acknowledgement

We wish to thank Jozef Širáň for his assitance in the preparation of this paper. The authors acknowledge support from the VEGA Research Grant 1/0781/11, the APVV Research Grant 0223-10, and the APVV support as part of the EUROCORES Programme EUROGIGA, project GREGAS, ESF-EC-0009-10, financed by the European Science Foundation.

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