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# Some results on Roman domination edge critical graphs

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**Abstract**

A Roman dominating function on a graph $G$ with vertex set $V(G)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number, $\gamma_R(G)$, of $G$ is the minimum weight of a Roman dominating function on $G$. In this paper we continue the study of Roman domination edge critical graphs by giving several properties and characterizations for these graphs.

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## 1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n$. We denote the *open neighborhood* of a vertex $v$ of $G$ by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v\in S} N(v)$, $N[S] = \bigcup_{v\in S} N[v]$, and $G[S]$ denotes the subgraph of $G$ induced by $S$. The *degree,* $deg(x)$, of a vertex $x$ denotes the number of its neighbors in $G$, and $\Delta(G)$ is the *maximum degree* of $G$. The *distance* between two vertices $x$ and $y$, denoted by $d_G(x, y)$, is the length of a shortest path from $x$ to $y$. The *diameter,* $\mathrm{diam}(G)$, of a graph $G$ is the maximum distance over all pairs of vertices of $G$. A set of vertices $S$ in $G$ is a *dominating set* if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. For notation and graph theory terminology in general we follow [5].

For a graph $G$, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0, V_1, V_2)$ of $V(G)$. So we will write $f = (V_0, V_1, V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on $G$. A function $f = (V_0, V_1, V_2)$ is called a $\gamma_R$*-function* (or $\gamma_R(G)$-function when we want to refer $f$ to $G$), if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. Roman domination has been introduced by Cockayne et al. [1].

It was observed in [3] that for every graph $G$ and an edge $e \notin E(G)$, $\gamma_R(G)-1 \leq \gamma_R(G+e) \leq \gamma_R(G)$. The same authors defined *Roman domination edge critical graphs*, or just $\gamma_R$*-edge critical graphs*, as graphs $G$ such that $\gamma_R(G+e) < \gamma_R(G)$ (i.e. $\gamma_R(G)-1 = \gamma_R(G+e)$) for any edge $e \notin E(G)$.

In this paper we continue the study of Roman domination edge critical graphs and obtain several properties and characterizations for these graphs. Recall that the *corona graph* $cor(G)$ of a graph $G$ is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added.

If $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function, then we say that any vertex $x \in V_1$ *Roman dominates* itself, and any vertex $y \in V_2$, Roman dominates $N[y]$.

## 2. Known results

We list in this section some known results that play an important role in our investigations. We begin by the following result that can be found in [1].

**Proposition 2.1.** [1]*Let* $f = (V_0; V_1; V_2)$ *be any* $\gamma_R(G)$*-function. Then*

*i) The subgraph induced by the vertices of* $V_1$ *has maximum degree one.*

*ii) No edge of* $G$ *joins* $V_1$ *to* $V_2$*.*

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, their *disjoint union* is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

**Proposition 2.2.** [1] *If a graph* $G$ *has no isolated vertex, then* $\gamma_R(G) = n$ *if and only if* $n$ *is even and* $G = \frac{n}{2}K_2$*.*

Hansberg, Jafari Rad and Volkmann [3] established a necessary and sufficient condition for $\gamma_R$-edge critical graphs. They also characterized several classes of $\gamma_R$-edge critical graphs including $\gamma_R$-edge critical trees and unicyclic graphs (see [4]).

**Proposition 2.3.** [3] *A graph* $G$ *is* $\gamma_R$*-edge critical if and only if for any two non-adjacent vertices* $x, y$*, there is a* $\gamma_R(G)$*-function* $f = (V_0, V_1, V_2)$ *such that* $\{f(x), f(y)\} = \{1, 2\}$*.*

Jafari Rad and Volkmann studied in [6] changing and unchanging the Roman domination number upon the removal of an arbitrary vertex or an edge. According to the effects of vertex removal on the Roman domination number of a graph $G$, let $V(G) = V_R^0 \cup V_R^+ \cup V_R^-$ such that

$$V_R^0 = \{v \in V(G) : \gamma_R(G - v) = \gamma_R(G)\},$$
$$V_R^+ = \{v \in V(G) : \gamma_R(G - v) > \gamma_R(G)\},$$
$$V_R^- = \{v \in V(G) : \gamma_R(G - v) < \gamma_R(G)\}.$$

**Proposition 2.4.** [6]*If* $v$ *is a vertex in a graph* $G$ *such that* $\gamma_R(G - v) > \gamma_R(G)$*, then* $f(v) = 2$ *for every* $\gamma_R(G)$*-function* $f$*.*

## 3. Some properties of $\gamma_R$-edge critical graphs

**Theorem 3.1.** *If* $G$ *is a* $\gamma_R$*-edge critical graph of order* $n \geq 2$*, then* $V(G) = V_R^- \cup V_R^0$*.*

*Proof.* Assume that $V_R^+ \neq \emptyset$ and let $x$ be any vertex of $V_R^+$. Then by Proposition 2.4, $f(x) = 2$ for every $\gamma_R(G)$-function. If $G[N(x)]$ is not complete, then let $a$ and $b$ be two non-adjacent vertices in $N(x)$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(a), f(b)\} = \{1, 2\}$, contradicting Proposition 2.1-ii. Thus $G[N(x)]$ is complete. Now let $u \in N(x)$. Since $V_R^+ \neq \emptyset$, we find that $G$ is not a complete graph. Thus $\gamma_R(G) > 2$, and so $u$ is not adjacent to all vertices of $G$. Let $v$ be any vertex not adjacent to $u$. Again by Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(u), f(v)\} = \{1, 2\}$. Since $f(x) = 2$, it follows that $f(u) = 2$ and $f(v) = 1$. But then the function $h$ defined on $V(G)$ by $h(x) = 0$ and $h(z) = f(z)$ if $z \neq x$ is an RDF for $G$ of weight less than $\gamma_R(G)$, a contradiction. Therefore $V_R^+ = \emptyset$. □

We note that the converse of Theorem 3.1 is not true. To see this consider a path $P_4$ with support vertices $u, v$ and leaves $u', v'$. Then $V_R^- = \{u', v'\}$ and $V_R^0 = \{u, v\}$ but $P_4$ is not $\gamma_R$-edge critical. On the other hand, we can wonder whether there are $\gamma_R$-edge critical graphs whose set $V_R^0$ is empty or not. The answer is given by the following two examples of graphs. The first one is the cycle $C_4$, where $\gamma_R(C_4) = 3$, $\gamma_R(C_4 + e) = 2$ for every edge $e \notin E(C_4)$ and $V_R^0 = \emptyset$. The second graph is the graph $G$ obtained from $cor(C_3)$ by subdividing two pendant edges. Clearly $G$ is $\gamma_R$-edge critical, and $\gamma_R(G) = 6$. Now if we consider any vertex $x$ of $G$ which is neither a leaf nor a support vertex, then $\gamma_R(G - x) = 6$, implying that $x \in V_R^0$ and so $V_R^0 \neq \emptyset$. Our next result shows that either $V_R^0$ is empty or induces a complete graph for every $\gamma_R$-edge critical graph.

**Theorem 3.2.** *If $G$ is a $\gamma_R$-edge critical graph, then either $V_R^0$ is empty or $G[V_R^0]$ is a complete graph.*

*Proof.* Let $G$ be a $\gamma_R$-edge critical graph. By Theorem 3.1, $V_R^+ = \emptyset$. Now assume that $V_R^0 \neq \emptyset$ and there are two non-adjacent vertices $x$ and $y$ in $V_R^0$. Then by Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(x), f(y)\} = \{1, 2\}$, say $f(x) = 1$. It follows that $x$ belongs to $V_R^-$, a contradiction. Hence if $V_R^0 \neq \emptyset$, then $G[V_R^0]$ is a complete graph. □

**Theorem 3.3.** *If $G$ is a $\gamma_R$-edge critical graph different from a complete graph, then $|V_R^-| \geq \gamma_R(G) - 2$. If $G$ is connected, then $|V_R^-| \geq \gamma_R(G) - 1$.*

*Proof.* Let $G$ be a $\gamma_R$-edge critical graph such that $G$ is not complete. By Theorem 3.1, $V(G) = V_R^- \cup V_R^0$. Clearly, if $V_R^0 = \emptyset$, then $|V_R^-| = |V(G)| \geq \gamma_R(G)$. Thus assume that $V_R^0 \neq \emptyset$, and let $w \in V_R^0$. According to Theorem 3.2, $G[V_R^0]$ is complete and therefore $(V_R^0 - \{w\}, V_R^-, \{w\})$ is an RDF for $G$ with weight $|V_R^-| + 2$ and thus $|V_R^-| \geq \gamma_R(G) - 2$.

If $G$ is connected, then let $u$ and $v$ be two adjacent vertices such that $u \in V_R^0$ and $v \in V_R^-$. Applying again Theorem 3.2, we see that $(\{v\} \cup V_R^0 - \{u\}, V_R^- - \{v\}, \{u\})$ is an RDF for $G$ with weight $|V_R^-| + 1$ and thus $|V_R^-| \geq \gamma_R(G) - 1$. □

If $H$ is the union of a $K_2$ and a $K_p$ for $p \geq 3$, then $\gamma_R(H) = 4$, $H$ is $\gamma_R$-edge critical and $|V_R^-| = 2$. Consequently, $|V_R^-| = \gamma_R(H) - 2$, and so the bound $|V_R^-| \geq \gamma_R(G) - 2$ in Theorem 3.3 is sharp.

We close this section by giving the following result. We omit the proof since it is similar to that used for Theorem 15 in [6].

**Theorem 3.4.** *Let $G$ be a $\gamma_R$-edge critical graph of order $n$ with $\gamma_R(G) > 2$. If $\gamma_R(G)$ is odd, then*

$$n \leq \frac{\gamma_R(G) - 1}{2}(\Delta(G) + 1) + 1, \tag{1}$$

*and if $\gamma_R(G)$ is even, then*

$$n \leq \frac{\gamma_R(G) - 2}{2}(\Delta(G) + 1) + 2. \tag{2}$$

## 4. $\gamma_R$-edge critical graphs for some values of $\gamma_R$

In this section, we will give characterizations of $\gamma_R$-edge critical graphs for some values of $\gamma_R$. For an integer $k \geq 2$, we call a graph $G$, $k$-$\gamma_R$-edge critical if $G$ is $\gamma_R$-edge critical and $\gamma_R(G) = k$. Since for any graph $G$ with at least two vertices, $\gamma_R(G) \geq 2$, the following characterization is obviously obtained.

**Theorem 4.1.** *A graph $G$ of order $n \geq 2$ is 2-$\gamma_R$-edge critical if and only if $G$ is a complete graph.*

Let $\mathcal{H}$ be the class of all graphs $G$ of order at least 3 such that $\Delta(G) = |V(G)| - 2$, and for any two non-adjacent vertices $x, y$ of $G$,

$$\Delta(G) \in \{deg(x), deg(y)\}.$$

**Theorem 4.2.** *A graph $G$ is 3-$\gamma_R$-edge critical if and only if $G \in \mathcal{H}$.*

*Proof.* Assume that $G$ is a 3-$\gamma_R$-edge critical graph of order $n \geq 3$, and let $h = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $V_2 = \emptyset$, then $|V_1| = |V(G)| = 3$. It follows that $G = K_1 \cup K_2$ and thus $G \in \mathcal{H}$. Next assume that $V_2 \neq \emptyset$. Then $|V_2| = 1$ and $|V_1| = 1$. Let $V_2 = \{x\}$ and $V_1 = \{y\}$. Since $\gamma_R(G) = 3$, it is obvious that $N[x] = V(G) - \{y\}$ and $|N[x]| \geq 2$. We consider the following cases.

**Case 1.** Assume that $G$ is disconnected. If $G[N[x]]$ is not a complete graph, then $\gamma_R(G + x'x'') > 2$ for every two non-adjacent vertices $x', x'' \in N(x)$, a contradiction. Thus $G[N[x]]$ is complete, implying that $G = K_{n-1} \cup K_1$. Hence $G \in \mathcal{H}$.

**Case 2.** Assume that $G$ is connected. We note that $N(y) \subseteq N(x)$ and no vertex in $N(y)$ is adjacent to all vertices of $N(x)$, for otherwise we assign 2 to such a vertex and 0 to the remaining vertices of $G$, to obtain an RDF for $G$, a contradiction. It follows that $G[N[x]]$ is not complete. We consider the following subcases. First assume that $N(x) = N(y)$. Let $a$ and $b$ be two arbitrary non-adjacent vertices in $N(x)$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(a), f(b)\} = \{1, 2\}$. Let $f(a) = 2$. Then $deg(a) = n - 2 = \Delta(G)$. Consequently, $G \in \mathcal{H}$.

Next assume that $N(x) \neq N(y)$. Then $N(x) - N(y) \neq \emptyset$. Let $a$ and $b$ be two non-adjacent vertices of $G$. If $a \in N(x) - N(y)$, and $b = y$, then by Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(a), f(y)\} = \{1, 2\}$, and clearly $f(a) = 2$. Then $a$ is adjacent to any vertex of $N(x) - \{a\}$, and so $deg(a) = \Delta(G) - 2$. Thus assume that $a, b \in N(x)$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f$ such that $\{f(a), f(b)\} = \{1, 2\}$. Let $f(a) = 2$. Then $a$ is adjacent to all vertices of $N(x) - \{a, b\}$. Consequently, $deg(a) = n - 2$ and $G \in \mathcal{H}$.

For the converse, it is straightforward to see that any graph in $\mathcal{H}$ is 3-$\gamma_R$-edge critical. □

**Theorem 4.3.** *A graph $G$ of order $n$ is $n$-$\gamma_R$-edge critical if and only if $n$ is even and $G = \frac{n}{2}K_2$ or $n$ is odd and $G = \frac{n-1}{2}K_2 \cup K_1$.*

*Proof.* It is easy to see that if $n$ is even and $G = \frac{n}{2}K_2$ or $n$ is odd and $G = \frac{n-1}{2}K_2 \cup K_1$, then $G$ is $n$-$\gamma_R$-edge critical.

Conversely, let $G$ be a $n$-$\gamma_R$-edge critical graph. If $G$ has no isolated vertex, then by Proposition 2.2, $n$ is even and $G = \frac{n}{2}K_2$. Thus we assume that $G$ has at least one isolated vertex. If $G$ has two isolated vertices $x, y$, then $f(x) = f(y) = 1$ for any $\gamma_R(G)$-function $f$. So $\gamma_R(G + xy) = \gamma_R(G)$, a contradiction. Hence $G$ has exactly one isolated vertex, say $x$, implying that $\gamma_R(G - x) = n - 1$. Again by Proposition 2.2, $n - 1$ is even and $G - \{x\} = \frac{n-1}{2}K_2$. We conclude that $n$ is odd and $G = \frac{n-1}{2}K_2 \cup K_1$. □

**Theorem 4.4.** *A graph $G$ of order $n \geq 3$ is $(n-1)$-$\gamma_R$-edge critical if and only if $G = C_i \cup m_1K_1 \cup m_2K_2$, where $i \in \{3, 4, 5\}$, $m_1 \leq 1$ and $m_1 + 2m_2 = n - i$.*

*Proof.* It is straightforward to see that if $G = C_i \cup m_1K_1 \cup m_2K_2$ with $i \in \{3, 4, 5\}$, $m_1 \leq 1$ and $m_1 + 2m_2 = n - i$, then $G$ is $(n-1)$-$\gamma_R$-edge critical.

Conversely, let $G$ be an $(n-1)$-$\gamma_R$-edge critical graph of order $n$. If $\Delta(G) \geq 3$, then $(N(v), V(G) - N[v], \{v\})$ is an RDF for $G$ with weight at most $n - 2$, where $v$ is a vertex of maximum degree. This contradiction implies that $\Delta(G) \leq 2$. So each component of $G$ is a path or a cycle.

Suppose to the contrary that no component of $G$ is a cycle. If $\Delta(G) \leq 1$, then $\gamma_R(G) = n$, a contradiction. Thus $\Delta(G) = 2$. Now let $P = v_1$-$v_2$-...-$v_k$ be a longest path in $G$. The condition $\Delta(G) = 2$ implies that $k \geq 3$. If $k \geq 6$, then $(\{v_1, v_3, v_4, v_6\}, V(G) - (N[v_2] \cup N[v_5]), \{v_2, v_5\})$ is an RDF for $G$ with weight $n - 2$, a contradiction. If $3 \leq k \leq 5$, then $\gamma_R(G + v_1v_k) = \gamma_R(G)$, a contradiction.

We deduce that at least one component of $G$ is a cycle. Let $C$ be a cycle of maximum order in $G$. If $|V(C)| \geq 6$, then there are two vertices $a, b$ on $C$ such that $N[a] \cap N[b] = \emptyset$, and so $(N(a) \cup N(b), V(G) - (N[a] \cup N[b]), \{a, b\})$ is an RDF for $G$ with weight $n - 2$ a contradiction. Thus $|V(C)| \leq 5$. Now let $G_1 = G - C$. If $\Delta(G_1) = 2$, then let $a \in V(G_1)$ be a vertex of degree 2 and $b \in V(C)$. Then $(N(a) \cup N(b), V(G) - (N[a] \cup N[b]), \{a, b\})$ is an RDF for $G$ of weight $n-2$, a contradiction. So $\Delta(G_1) \leq 1$, implying that $C$ is unique cycle in $G$ and the remaining components are either $K_2$ or $K_1$. Therefore $G = C_i \cup m_1K_1 \cup m_2K_2$, where $i \in \{3, 4, 5\}$ and $m_1 + 2m_2 = n - i$. If $m_1 \geq 2$, then adding any edge between two isolated vertices does not decrease the roman domination number, which is a contradiction. Thus $m_1 \leq 1$. This completes the proof. □

We close this section by mentioning the following remark. Recall first that a graph $G$ is *$\gamma$-edge critical*, if $\gamma(G + e) < \gamma(G)$ for every $e \notin E(G)$. (See for example [2, 7]). A $\gamma_R$-edge critical graph is not necessarily $\gamma$-edge critical, as can be seen by the cycle $C_5$. Also a $\gamma$-edge critical graph is not necessarily $\gamma_R$-edge critical, as can be seen by the complement of a complete graph of order at least two.

## 5. Diameter of $\gamma_R$-edge critical graphs

**Theorem 5.1.** *If $G$ is a $\gamma_R$-edge critical connected graph with $\gamma_R(G) > 3$, then* $\operatorname{diam}(G) \leq 3\lceil \frac{\gamma_R(G)-3}{2} \rceil + 2$.

*Proof.* Assume $G$ is a $\gamma_R$-edge critical graph such that $\gamma_R(G) > 3$. Let $x$ and $y$ be two vertices with $d_G(x, y) = \operatorname{diam}(G) = d$. For $i = 0, 1, 2, ..., d$, let $A_i$ be the set of all vertices of $G$ at distance $i$ from $x$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x), f(y)\} = \{1, 2\}$. Without loss of generality, assume that $f(y) = 2$. Then $y$ Roman dominates at most the vertices in $A_d \cup A_{d-1}$, and $x$ Roman dominates only $A_0$. Also any vertex of $V_2$ belonging to some $A_i$, with $i \notin \{0, d-1, d\}$, Roman dominates at most the vertices of $A_{i-1} \cup A_i \cup A_{i+1}$. Let $d = 3i + r + 2$, where $r = 0, 1$ or $2$. If $r = 0$, then $\gamma_R(G) = f(V_1 \cup V_2) \geq 2i + 3$. If $r = 1$, then $\gamma_R(G) = f(V_1 \cup V_2) \geq 2i + 4$, and if $r = 2$, then $\gamma_R(G) = f(V_1 \cup V_2) \geq 2i + 5$. Therefore the desired bound follows. □

A *matching* in a graph $G$ is a subset of pair-wise non-incident edges. A matching $M$ is said to be *perfect* if $|M| = |V(G)|/2$. Let $\mathcal{F}$ be the class of all graphs $G$ that are obtained by first joining every vertex of a complete graph $K_m$ for some $m \geq 2$ to every vertex of $K_n - M$ for an even $n \geq 2$, where $M$ is a perfect matching of $K_n$, and then adding a path $P_2$ by joining one of its end vertices to every vertex in $K_n - M$. Our next result improves the upper bound in Theorem 5.1 for $\gamma_R$-edge critical graphs $G$ with $\gamma_R(G) = 4$.

**Theorem 5.2.** *If $G$ is a 4-$\gamma_R$-edge critical connected graph, then* $\operatorname{diam}(G) \leq 3$*, with equality if and only if $G \in \mathcal{F}$.*

*Proof.* Assume that $G$ is a $\gamma_R$-edge critical graph such that $\gamma_R(G) = 4$. By Theorem 5.1, $\operatorname{diam}(G) \leq 5$. Let $x$ and $y$ be two vertices with $d_G(x, y) = \operatorname{diam}(G) = d$. For $i = 0, 1, 2, ..., d$, let $A_i$ be the set of all vertices of $G$ at distance $i$ from $x$. Assume that $\operatorname{diam}(G) = 5$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x), f(y)\} = \{1, 2\}$. Without loss of generality assume that $f(x) = 2$. Then $A_3 \cup A_4 \cup A_5$ is not Roman dominated, a contradiction. Thus $\operatorname{diam}(G) \leq 4$. Assume that $\operatorname{diam}(G) = 4$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that, without loss of generality, $f(x) = 1$ and $f(y) = 2$. Then $y$ Roman dominates at most the vertices in $A_4 \cup A_3$ and $x$ Roman dominates only $A_0$. Now since $\gamma_R(G) = 4$, we must have a vertex $A_1 \cup A_2$ assigned 1 and such a vertex cannot Roman dominate all vertices in $A_1 \cup A_2$, a contradiction. It follows that $\operatorname{diam}(G) \leq 3$.

Now assume that $G$ is a 4-$\gamma_R$-edge critical graph with $\operatorname{diam}(G) = 3$. Let $P = x\text{-}a\text{-}b\text{-}y$ be a diametrical path in $G$. By Proposition 2.3, and without loss of generality, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $f(x) = 2$ and $f(y) = 1$. Then $f(b) = 1$ and $f(a) = 0$. If $y$ has degree at least two, then one of its neighbors is assigned 0 implying that $d(x, y) = 2$, a contradiction. Hence $\deg(y) = 1$. Note that $b$ is the unique vertex at distance two from $x$ for otherwise such a vertex will be assigned 0 and has no neighbor assigned 2. Now if $\deg(x) = 1$, then $G = P_4$, contradicting the fact that $\gamma_R(G) = 4$.

Thus $\deg(x) \geq 2$. If $N(x) \subseteq N(b)$, then $(N(b), \{x\}, \{b\})$ is an RDF for $G$ of weight 3, a contradiction. Thus $N(x) \nsubseteq N(b)$. Let $z \in N(x) - N(b)$. By Proposition 2.3, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $\{f(z), f(y)\} = \{1, 2\}$. We obtain that $f(z) = 2$. It follows that $z$ is adjacent to every vertex in $N(x)$. We deduce the same for every vertex in $N(x) - N(b)$. Thus any vertex of $N(x) - N(b)$ is adjacent to all vertices of $N(x)$. In particular, the subgraph induced by $N[x] - N(b)$ is complete of order at least two. Now let $w$ be an arbitrary vertex in $N(x) \cap N(b)$. Again by Proposition 2.3, there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ such that $\{f(w), f(y)\} = \{1, 2\}$. We can see that $f(w) = 2$. Since $w$ is adjacent to all vertices of $N(x) - N(b)$, and $\gamma_R(G) = 4$, we obtain that $w$ is adjacent to all vertices $N(x)$ except one, say $w^*$, where $w^* \in N(x) \cap N(b)$. Since $w$ is chosen arbitrarily in $N(x) \cap N(b)$, we deduce that $|N(x) \cap N(b)|$ is even and the subgraph induced by $N(x) \cap N(b)$ is a complete graph minus a perfect matching $M$. Therefore $G \in \mathcal{F}$.

For the converse, it is straightforward to see that if $G \in \mathcal{F}$, then $G$ is a 4-$\gamma_R$-edge critical graph with diameter 3. □

**Theorem 5.3.** *For every even integer $n \geq 6$, there is an $n$-$\gamma_R$-edge critical graph $G$ with diam$(G) = 5$.*

*Proof.* Let $n \geq 6$ be an even integer and let $m = \frac{n}{2}$. Consider the graph $cor(K_m)$ with vertex set $\{v_1, v_2, ..., v_m, u_1, u_2, ..., u_m\}$, where each $v_i$ belongs to $K_m$ and each $u_i$ is a leaf. Let $G_m$ be the graph obtained from $cor(K_m)$ by subdividing the first $m-1$ pendant edges exactly once. Let $w_1, w_2, ..., w_{m-1}$ be the new vertices, where $N(w_i) = \{v_i, u_i\}$ for each $i = 1, 2, ..., (m-1)$. Clearly $\text{diam}(G_m) = 5$. We first show that $\gamma_R(G_m) = 2m = n$. It is obvious that $(V(G) - \{w_1, w_2, ..., w_{m-1}, v_m\}, \emptyset, \{w_1, w_2, ..., w_{m-1}, v_m\})$ is an RDF for $G_m$, and so $\gamma_R(G_m) \leq 2m = n$. Next we show that $\gamma_R(G_m) \geq 2m$. Let $f$ be a $\gamma_R(G_m)$-function. Observe that for every $i = 1, 2, ..., (m-1)$ we have $f(v_i) + f(u_i) + f(w_i) \geq 2$. The inequality is strict if $f(v_i) \neq 0$. Clearly, $f(V(G) - \{v_m, u_m\}) \geq 2m - 2$. Now if there is some $i \neq m$ such that $f(v_i) \neq 0$, then $f(v_i) + f(u_i) + f(w_i) \geq 3$ and $f(u_m) + f(v_m) \geq 1$, implying that $f(V(G)) \geq 2m$. Thus we may assume that for every $i \neq m$, $f(v_i) = 0$. It follows that $f(u_m) + f(v_m) = 2$ and hence $f(V(G)) \geq 2m$. In any case, $\gamma_R(G_m) = f(V(G)) \geq 2m$, implying that $\gamma_R(G_m) = 2m = n$. Now it is straightforward to see that $G_m$ is $\gamma_R$-edge critical. □

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