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The distance Roman domatic number of a graph

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Abstract

Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A *k-distance Roman dominating function* on G is a labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a vertex with label 2 within distance k from each other. A set $\{f_1, f_2, \ldots, f_d\}$ of k-distance Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(G)$, is called a *k-distance Roman dominating family* (of functions) on G. The maximum number of functions in a k-distance Roman dominating family on G is the *k-distance Roman domatic number* of G, denoted by $d_R^k(G)$. In this paper we initiate the study of k-distance Roman domatic number in graphs and we present some sharp bounds for $d_R^k(G)$. In addition, we determine the k-distance Roman domatic number of some graphs.

Keywords: distance Roman dominating function, distance Roman domination number, distance Roman domatic number, Roman domination number, Roman domatic number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. Denote by K_n the *complete graph*, by C_n the *cycle* and by P_n the *path* of order n, respectively. The complement of a graph G is denoted by $\overline{G}$. Given two graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, the *disjoint union* is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let k

be a positive integer. For two vertices x and y, let $d(x,y)$ denote the distance between x and y in G. The *girth* $g(G)$ of a graph G is the length of its shortest cycle. For a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u,v) \leq k\}$ and the *closed k-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open k-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$, and the *closed neighborhood* $N_{k,G}[S]$ of S is the set $N_{k,G}(S) \cup S$. The *k-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph G is called *distance-k-regular*. The *k-th power* G^k of a graph G is the graph with vertex set $V(G)$ where two different vertices u and v are adjacent if and only if the distance $d(u,v)$ is at most k in G. Now we observe that $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $\deg_{k,G}(v) = \deg_{1,G^k}(v) = \deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. Consult [8] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer. A set $D \subseteq V(G)$ is a k-distance dominating set of G if every vertex in $V(G) - D$ is within distance k of at least one vertex in D. The k-distance domination number $\gamma^k(G)$ of G is the minimum cardinality among all k-distance dominating sets of G. A k-distance domatic partition is a partition of V into k-distance dominating sets, and the k-distance domatic number $d^k(G)$ is the largest number of sets in a k-distance domatic partition. The concept of k-distance domatic number was introduced by Zelinka in [9]. It is clear that

$$\gamma^k(G) \cdot d^k(G) \leq n. \tag{1}$$

A *k-distance Roman dominating function* (kDRDF) on a graph $G = (V,E)$ is a function $f : V \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ has a vertex u for which $f(u) = 2$ and $d(u,v) \leq k$. The *weight* of an kDRDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *k-distance Roman domination number* of a graph G (see [1]), denoted by $\gamma_R^k(G)$, equals the minimum weight of an kDRDF on G. A $\gamma_R^k(G)$*-function* is a k-distance Roman dominating function of G with weight $\gamma_R^k(G)$. By these definitions, we easily obtain

$$\gamma_R^k(G) = \gamma_R(G^k). \tag{2}$$

A k-distance Roman dominating function $f : V \longrightarrow \{0,1,2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a k-distance dominating set when f is an kDRDF, and since placing weight 2 at the vertices of a k-distance dominating set yields an kDRDF, we have

$$\gamma^k(G) \leq \gamma_R^k(G) \leq 2\gamma^k(G). \tag{3}$$

Note that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual *Roman domination number* $\gamma_R(G)$. The definition of the Roman dominating function was given multiplicity by Steward [7] and ReVelle and Rosing [5]. Cockayne et al. [4] as well as Chambers et al. [3] have given a lot of results on Roman domination.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct k-distance Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(G)$, is called a *k-distance Roman dominating family* (of functions) on G. The maximum number of functions in a k-distance Roman dominating family (kDRD family) on G is the *k-distance Roman domatic number* of G, denoted by $d_R^k(G)$. In the case $k = 1$ we write $d_R(G)$ instead of $d_R^1(G)$. The parameter $d_R(G)$ was introduced and investigated in [6]. The k-distance Roman domatic number is well-defined and

$$d_R^k(G) \geq 1 \tag{4}$$

for all graphs G since the set consisting of any kDRDF forms an kDRD family on G. Obviously,

$$d_R^k(G) = d_R(G^k). \tag{5}$$

If $G_1, G_2, \cdots, G_s$ are the components of G, then $d_R^k(G) = \min\{d_R^k(G_1), d_R^k(G_2), \ldots, d_R^k(G_s)\}$. Thus we consider only connected graphs.

Our purpose in this paper is to initiate the study of k-distance Roman domatic numbers in graphs. We first study basic properties and bounds for the k-distance Roman domatic number of a graph. In addition, we determine the k-distance Roman domatic number of some classes of graphs.

We start with the following observations and properties.

Observation 1. *Let k and l be two positive integers such that $k < l$. If G is a graph, then $d_R^k(G) \leq d_R^l(G)$.*

Proof. By the definition of the k-distance Roman dominating function it is clear that each k-distance Roman dominating function on G is also an l-distance Roman dominating function on G. Therefore each k-distance dominating family of G is an l-distance dominating family of G, and this implies $d_R^k(G) \leq d_R^l(G)$. □

Observation 2. *Let K_n be the complete graph of order n and k a positive integer. Then*

$$d_R^k(K_n) = n.$$

Observation 3. *If G is a graph, then $d_R^k(G) = 1$ if and only if G is empty.*

Observation 4. *Let $k \geq 1$ be an integer, and let G be a graph of order n. If the diameter* $\operatorname{diam}(G) \leq k$*, then $\gamma_R^k(G) = \gamma_R(K_n)$ and $d_R^k(G) = d_R(K_n)$.*

The next result is an immediate consequences of Observations 2 and 4.

Corollary 5. *If $k \geq 2$ and G is a graph of order n with* $\operatorname{diam}(G) = 2$*, then $\gamma_R^k(G) = 2$ and $d_R^k(G) = n$.*

Corollary 6. *Let $k \geq 2$ be an integer, and let G be a graph of order n. If* $\operatorname{diam}(G) \neq 3$*, then $\gamma_R^k(G) = \gamma_R(K_n)$ and $d_R^k(G) = d_R(K_n)$ or $\gamma_R^k(\overline{G}) = \gamma_R(K_n)$ and $d_R^k(\overline{G}) = d_R(K_n)$.*

Proof. If $\operatorname{diam}(G) \le 2$, then it follows from Observation 4 that $\gamma_R^k(G) = \gamma_R(K_n)$ and $d_R^k(G) = d_R(K_n)$. If $\operatorname{diam}(G) \ge 3$, then the hypothesis $\operatorname{diam}(G) \ne 3$ implies that $\operatorname{diam}(G) \ge 4$. Now, according to a result of Bondy and Murty [2] (page 14), we deduce that $\operatorname{diam}(\overline{G}) \le 2$. Applying again Observation 4, we obtain $\gamma_R^k(\overline{G}) = \gamma_R(K_n)$ and $d_R^k(\overline{G}) = d_R(K_n)$. □

Corollary 7. *If $k \ge 3$ is an integer and G a graph of order n, then $\gamma_R^k(G) = \gamma_R(K_n)$ and $d_R^k(G) = d_R(K_n)$ or $\gamma_R^k(\overline{G}) = \gamma_R(K_n)$ and $d_R^k(\overline{G}) = d_R(K_n)$.*

We make use of the following results in this paper.

Proposition A. [1] *For $n \ge 3$,*

$$\gamma_R^k(C_n) = \begin{cases} 2\lfloor \frac{n}{2k+1} \rfloor + 1 & n \equiv 1 \pmod{2k+1} \\ 2\lceil \frac{n}{2k+1} \rceil & \text{otherwise.} \end{cases}$$

Proposition B. [9] *Let G be a connected graph of order n and let k be a positive integer. Then*

$$d^k(G) \ge \min\{n, k+1\}.$$

Proposition C. [9] *For $n \ge 3$, $\gamma^k(C_n) = \lceil \frac{n}{2k+1} \rceil$ and $d^k(C_n) = \lfloor \frac{n}{\lceil \frac{n}{2k+1} \rceil} \rfloor$.*

Proposition D. [6] *Let G be a graph of order $n \ge 2$. Then $\gamma_R(G) = n$ and $d_R(G) = 2$ if and only if $\Delta(G) = 1$.*

2. Properties of the k-distance Roman domatic number

In this section we present basic properties of $d_R^k(G)$ and sharp bounds on the k-distance Roman domatic number of a graph.

Theorem 8. *For every graph G,*

$$d_R^k(G) \le \delta_k(G) + 2.$$

Moreover, if $d_R^k(G) = \delta_k(G) + 2$, then for each function of any d_R^k–family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of minimum k-degree $\delta_k(G)$, $\sum_{u \in N_{k,G}(v)} f_i(u) = 1$ for exactly two f_is, say f_1, f_2, and $\sum_{u \in N_{k,G}(v)} f_i(u) = 2$ for $i \ge 3$ and $\sum_{i=1}^d f_i(u) = 2$ for every $u \in N_{k,G}(v)$.

Proof. If $d_R^k(G) \le 2$, the result is immediate. Let now $d_R^k(G) \ge 3$ and let $\{f_1, f_2, \ldots, f_d\}$ be a kDRD family on G such that $d = d_R^k(G)$. Assume that v is a vertex of minimum k-degree $\delta_k(G)$. Since the equality $\sum_{u \in N_{k,G}[v]} f_i(u) = 1$ holds for at most two indices $i \in \{1, 2, \ldots, d\}$, we have

$$\begin{aligned} 2d - 2 &\le \textstyle\sum_{i=1}^d \sum_{u \in N_{k,G}[v]} f_i(u) \\ &= \textstyle\sum_{u \in N_{k,G}[v]} \sum_{i=1}^d f_i(u) \\ &\le \textstyle\sum_{u \in N_{k,G}[v]} 2 \\ &= 2(\delta_k(G) + 1). \end{aligned}$$

Thus $d_R^k(G) \le \delta_k(G)+2$.

If $d_R^k(G) = \delta_k(G)+2$, then the two inequalities occurring in the proof become equalities, and the desired result follows. $\square$

Theorem 9. *Let G be a graph of order n with k-distance Roman domination number $\gamma_R^k(G)$ and k-distance Roman domatic number $d_R^k(G)$. Then*

$$\gamma_R^k(G) \cdot d_R^k(G) \le 2n.$$

Moreover, if $\gamma_R^k(G) \cdot d_R^k(G) = 2n$, then for each kDRD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_R^k(G)$, each function f_i is a $\gamma_R^k(G)$-function and $\sum_{i=1}^d f_i(v) = 2$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a kDRD family on G such that $d = d_R^k(G)$ and let $v \in V$. Then

$$\begin{array}{rcl} d \cdot \gamma_R^k(G) & = & \sum_{i=1}^d \gamma_R^k(G) \\ & \le & \sum_{i=1}^d \sum_{v\in V} f_i(v) \\ & = & \sum_{v\in V} \sum_{i=1}^d f_i(v) \\ & \le & \sum_{v\in V} 2 \\ & = & 2n. \end{array}$$

If $\gamma_R^k(G) \cdot d_R^k(G) = 2n$, then the two inequalities occurring in the proof become equalities. Hence for the kDRD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v\in V} f_i(v) = \gamma_R^k(G)$, thus each function f_i is a $\gamma_R^k(G)$-function, and $\sum_{i=1}^d f_i(v) = 2$ for all $v \in V$. $\square$

Let $A_1 \cup A_2 \cup \ldots \cup A_d$ be a k-distance domatic partition of $V(G)$ into k-distance dominating sets such that $d = d^k(G)$. Then the set of functions $\{f_1, f_2, \ldots, f_d\}$ with $f_i(v) = 2$ if $v \in A_i$ and $f_i(v) = 0$ otherwise for $1 \le i \le d$ is an kDRD family on G. This shows that $d^k(G) \le d_R^k(G)$ for every graph G. Since $\gamma_R^k(G) \ge 2$ for each graph G of order $n \ge 2$, Theorem 9 implies that $d_R^k(G) \le n$. Combining these two observations and Proposition B, we obtain the following results.

Corollary 10. *For any graph G of order $n \ge 2$, $d^k(G) \le d_R^k(G) \le n$.*

Corollary 11. *For any graph G of order $n \ge 2$ with $\gamma_R^k(G) = 2\gamma^k(G)$ and $d^k(G) = \lfloor \frac{n}{\gamma^k(G)} \rfloor$, $d_R^k(G) = d^k(G)$.*

Proof. Let G be a graph of order $n \ge 2$ with $\gamma_R^k(G) = 2\gamma^k(G)$ and $d^k(G) = \lfloor \frac{n}{\gamma^k(G)} \rfloor$. It follows from Corollary 10 and Theorem 9 that

$$d^k(G) \le d_R^k(G) \le \lfloor \frac{2n}{\gamma_R^k(G)} \rfloor = \lfloor \frac{n}{\gamma^k(G)} \rfloor = d^k(G),$$

as desired. $\square$

Corollary 12. *If G is a connected graph of order n, then $d_R^k(G) \geq \min\{n, k+1\}$.*

Proposition 13. *Let n, k be two positive integers such that $n \geq k+2$. Then $d_R^k(P_n) = k+1$.*

Proof. Let $P_n := v_1 \dots v_n$. Then obviously $N_{k,P_n}(v_1) = \{v_2, \dots, v_{k+1}\}$. By Theorem 8, $d_R^k(P_n) \leq |N_{k,P_n}(v_1)| + 2 = k+2$. By Corollary 12, it is sufficient to show that $d_R^k(P_n) \leq k+1$. Assume to the contrary that $d_R^k(P_n) = k+2$ and suppose $\{f_1, f_2, \dots, f_d\}$ is an kDRD family on P_n such that $d = d_R^k(P_n)$. By Theorem 8 we may assume that $\sum_{u \in N_{k,G}(v_1)} f_i(u) = 1$ for $i = 1, 2$, and $\sum_{u \in N_{k,G}(v_1)} f_i(u) = 2$ for $i \geq 3$ and $\sum_{i=1}^{d} f_i(u) = 2$ for every $u \in N_{k,G}(v_1)$. Since $\sum_{u \in N_{k,G}(v_1)} f_i(u) = 1$ and f_i is a kDRDF on P_n for $i = 1, 2$, we must have $f_1(v_{k+2}) = f_2(v_{k+2}) = 2$ implying that $\sum_{i=1}^{d} f_i(v_{k+2}) \geq 4$ which is a contradiction. Thus $d_R^k(P_n) \leq k+1$ and the proof is complete. □

Using Theorems A, C, and Corollary 11, we determine the distance Roman domatic number of some cycles.

Proposition 14. *If C_n is the cycle on $n \geq 3$ vertices and $n \not\equiv 1 \pmod{2k+1}$, then*

$$d_R^k(C_n) = \lfloor \frac{n}{\lceil \frac{n}{2k+1} \rceil} \rfloor.$$

Proof. It follows from Theorems A and C that $\gamma_R^k(G) = 2\gamma^k(G)$ and $d^k(G) = \lfloor \frac{n}{\gamma^k(G)} \rfloor$ and the result follows from Corollary 11. □

Proposition 15. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_R^k(G) = n$ and $d_R^k(G) = 2$ if and only if $G = K_2$.*

Proof. Let $d_R^k(G) = 2$. Then Corollary 12 and the fact $n \geq 2$ implies that $2 = \min\{n, k+1\}$. If $n = 2$, then it follows from Observation 3 that $G = K_2$. Let $k+1 = 2$. Then $k = 1$ and the result follows from Proposition D.

Conversely, let $G = K_2$. Then obviously $\gamma_R^k(G) = d_R^k(G) = 2$. □

Proposition 16. *If G is a connected graph of order $n \geq 2$, then $d_R^k(G) = n$ if and only if G^k is the complete graph on n vertices.*

Proof. If G^k is the complete graph on n vertices, then the result follows by (5) and Observation 2.

Conversely, let $d_R^k(G) = n$. Then $n \leq k+1$ by Corollary 12. This implies that $\operatorname{diam}(G) \leq k$ and hence G^k is the complete graph. □

Theorem 17. *If G is a connected graph of order $n \geq 2$, then*

$$\gamma_R^k(G) + d_R^k(G) \leq n + 2 \tag{6}$$

with equality if and only if G^k is a complete graph.

Proof. If $d_R^k(G) = 1$, then obviously $\gamma_R^k(G) + d_R^k(G) \leq n + 1$. Let now $d_R^k(G) \geq 2$. Since $\gamma_R^k(G) \geq 2$, we have $d_R^k(G) \leq n$. According to Theorem 9, we obtain

$$\gamma_R^k(G) + d_R^k(G) \leq \frac{2n}{d_R^k(G)} + d_R^k(G). \tag{7}$$

Using the fact that the function $g(x) = x + (2n)/x$ is decreasing for $2 \leq x \leq \sqrt{2n}$ and increasing for $\sqrt{2n} \leq x \leq n$, this inequality leads to the desired bound immediately.

If G^k is the complete graph on n vertices, then obviously $\gamma_R^k(G) = 2$ and by Observation 2 and (5), $d_R^k(G) = n$. Thus $\gamma_R^k(G) + d_R^k(G) = n + 2$.

Conversely, let equality hold in (6). It follows from (7) that

$$n + 2 = \gamma_R^k(G) + d_R^k(G) \leq \frac{2n}{d_R^k(G)} + d_R^k(G) \leq n + 2,$$

which implies that $\gamma_R^k(G) = \frac{2n}{d_R^k(G)}$ and $d_R^k(G) = 2$ or $d_R^k(G) = n$. If $d_R^k(G) = n$, then G^k is a complete graph by Proposition 16. If $d_R^k(G) = 2$, then $\gamma_R^k(G) = n$, and it follows from Proposition 15 that $G = K_2$. This completes the proof. □

Corollary 18. *Let $k \geq 1$ be an integer, and let G be a connected graph of order $n \geq 2$. If $\min\{\gamma_R^k(G), d_R^k(G)\} \geq a$ with $3 \leq a \leq \sqrt{2n}$, then*

$$\gamma_R^k(G) + d_R^k(G) \leq a + \frac{2n}{a}.$$

Proof. Since $\min\{\gamma_R^k(G), d_R^k(G)\} \geq a \geq 3$, it follows from Theorem 9 that $a \leq d_R^k(G) \leq \frac{2n}{a}$. Applying inequality (7), we obtain

$$\gamma_R^k(G) + d_R^k(G) \leq \frac{2n}{d_R^k(G)} + d_R^k(G) \leq a + \frac{2n}{a}.$$

□

Corollary 10 implies the following Nordhaus-Gaddum type result immediately.

Corollary 19. *If G is a graph of order $n \geq 2$, then $d_R^k(G) + d_R^k(\overline{G}) \leq 2n$.*

If G^k and $\overline{G}^k$ are complete graphs, then Observation 2 and (5) yield to $d_R^k(G) + d_R^k(\overline{G}) = 2n$. This demonstrates that the above Nordhaus-Gaddum inequality is sharp.

Observation 20. *Let $k \geq 3$ be an integer, and let G be a graph of order $n \geq 2$. If neither G nor $\overline{G}$ is empty, then $d_R^k(G) + d_R^k(\overline{G}) \geq n + 2$.*

Proof. Since $k \geq 3$, it follows from Corollary 7 that $d_R^k(G) = d_R(K_n) = n$ or $d_R^k(\overline{G}) = n$. Assume, without loss of generality, that $d_R^k(G) = n$. As $\overline{G}$ is not empty, we deduce from Observation 3 that $d_R^k(\overline{G}) \geq 2$. This leads to the desired bound $d_R^k(G)+d_R^k(\overline{G}) \geq n+2$. □

We conclude this paper with an open problem.

Problem. Prove or disprove: If C_n is the cycle on $n \geq 3$ vertices and $n \equiv 1 \pmod{2k+1}$, then

$$d_R^k(C_n) = \lfloor \frac{n}{\lceil \frac{n}{2k+1} \rceil} \rfloor.$$

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